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Appendix VII. The influence of boundary conditions on a probabilistic evolution governed by time symmetrical laws

Abstract

Different hypotheses for the origin of the time asymmetries in the universe are described. A law, proposed by the present author, is taken as a point of departure to show how constraints on the boundary conditions in one or both temporal ends of the universe constitutes this origin. Support for this hypothesis is found in the Ehrenfest urn model.

1 Introduction

In our universe, various phenomena demonstrate differences between past and future time direction. Overall expansion of the universe, life-cycle of stars, eroding cliffs, biological processes, and heat dissipation are disparate examples of time asymmetric processes at different levels of magnitude. At the most fundamental level, a difference between the two directions of time becomes evident in quantum physics. The equations of quantum mechanics can, following Born's rule, successfully describe the probabilities of various outcomes, given a *preceding* measurement, whereas the probabilities of outcomes given a *succeeding* measurement are not generally described by these equations.[1]

Whether these time asymmetries have common or different origins is still an open question and several origins have been proposed (in order of first documented appearance found by the present author):

A. The universe as a whole is in a state of equilibrium, but the present human civilization lives in a region that, through a random fluctuation, happens to be in an extreme non-equilibrium state.[2]

[1] Among others, Fock (1948), Watanabe (1955), Landau & Lifshitz (1958), Penrose (1989), Gell-Mann & Hartle (1994), and Bohm et al. (2011) have pointed out that the equations of quantum mechanics demonstrates different degrees of applicability with regard to the two directions of time. See also Skoruppa (2022b).

[2] Boltzmann (1897).

B. Time asymmetries have their origin in the extreme initial boundary condition of the universe with regard to, for example, entropy, disequilibrium, density, smoothness, and expansion rate.[3]

C. Quantum mechanics is associated with a fundamental difference between the past and future time directions, from which most other time asymmetries can be derived.[4]

D. The time asymmetry of probabilistic laws in, for example, genetics, statistical physics, and quantum mechanics has its source in a time asymmetry concerning causality.[5]

E. The time asymmetries of the universe only exist from a human perspective.[6]

F. The time asymmetries in our world are an effect of the expansion of the universe.[7]

G. While the past time end of the universe in some sense is constrained within narrow limits, the future destiny of the universe is a random consequence of the quantum mechanical laws applied to the past boundary condition.[8]

H. The decay of the neutral K-meson is an expression of the most fundamental time asymmetry, from which other time asymmetries can be inferred.[9]

No consensus has been reached in the scientific community with respect to which of the hypotheses above is correct and the problem can still be considered unsolved. In the present article, a combination of hypotheses B and G is proposed as an explanation for the connection

[3]This explanation was proposed already by Ludwig Boltzmann (1897), long before the Big Bang theory emerged. However, Boltzmann preferred hypothesis A. More recently, the idea was advocated by Cocke (1967), who proposed a time-symmetric universe with both an expanding and a contracting phase, where the time asymmetries have opposite directions. Cocke analyses a modified Ehrenfest urn model in a similar way as it is done in the present article, but he arrives, however, at other conclusions.

[4]Born (1948) made an attempt to derive the time asymmetry of thermodynamics from quantum theory. Gell-Mann & Hartle (1994) describe the connection between the time asymmetric application of the quantum mechanical equations and other physical time asymmetries, though not as an explanation.

[5]Reichenbach (1956), Sober (1993).

[6]Mehlberg (1961), Watanabe (1966).

[7]Gold (1962), more recently elaborated by Carroll & Chen (2004) on the basis of eternal inflation.

[8]Gell-Mann & Hartle (1994).

[9]Cramer (2000, p. 184) asserts with reference to the K-meson decay: “It is generally agreed that the CP violation arrow is probably the most fundamental time arrow [...]”.

between the expansion of the universe and other physical time asymmetries.[10]

The outline of the article is as follows. In Section 2, a physical law, aimed at giving a general description of time asymmetries in our universe, is proposed. Then, some basic assumptions and requirements for a suitable model of the universe are presented in Section 3. In Section 4, a modified version of Ehrenfest's urn experiment is described as a model of the universe, and the conditions for time symmetry are established. The model is supplemented in Section 5, by allowing only the trajectories that contain a special initial boundary condition; it is furthermore shown that a random choice among these trajectories represents an evolution in accordance with the law proposed in Section 2. In Section 6, the conclusions are exemplified in a two-ball Ehrenfest model, and in Section 7 the expected evolution of this particular model is described, given two different kinds of boundary conditions. Finally, in Section 8, the results are discussed.

2 Statistical time asymmetry

To establish a cosmological origin of the different physical time asymmetries, these asymmetries must be coherently formulated. Such a formulation has been previously suggested to have its foundation in the concept of entropy, but objections can be raised against this proposal. The most important of these is the lack of generality of the entropy concept, which only has been given a clear and empirically valid definition for a limited range of systems in equilibrium.[11]

In Skoruppa (2022d), an alternative general description of a range of various time asymmetries is proposed in the form of a law (here with the states defined at points of time instead of during time intervals – the latter alternative is used in the referred article):

> **Law of general statistical time asymmetry**
> Assume that $A[t_1]$ and $B[t_2]$, where $t_1 < t_2$, are physical states, defined by the same quantities in a process that is uncontrolled in the time interval $[t_1, t_2]$, and where only one of the non-deterministic probabilities $p(B[t_2]|A[t_1])$ and $p(A[t_1]|B[t_2])$ is time translation invariant. Then the former probability is always time translation invariant.

The word "uncontrolled" here means that no manipulation is made during the process to control the outcome of the process in either time direction, which is a more loose condition than isolation.

[10]Such a proposal is in accordance with Gell-Mann & Hartle (1994).

[11]Earman (1974, 2006), Denbigh (1989), and Uffink (2007) point out the lack of generality of the entropy concept. See also Skoruppa (2022c) for further details.

The two kinds of conditional probabilities, involved in this law, can be defined in the following way:

Definition 1 *Future-directed probability:*

$$p(B[t_2]|A[t_1]) = \frac{p(A[t_1] \wedge B[t_2])}{p(A[t_1])} ,$$

where $t_1 < t_2$.

Definition 2 *Past-directed probability:*

$$p(A[t_1]|B[t_2]) = \frac{p(A[t_1] \wedge B[t_2])}{p(B[t_2])} ,$$

where $t_1 < t_2$.

In more ordinary language, the *law of general statistical time asymmetry* can be expressed in the following way: in a process that can be correctly described by either future-directed probabilities or past-directed probabilities that are lawlike (i.e. time translation invariant), but not both, lawlike future-directed probabilities *always* give the correct description. This law, mostly in one of its more specialized versions, *the law of statistical time asymmetry* and *the law of statistical mechanical time asymmetry*, has been verified in Skoruppa (2022a,b,d) for a number of physical processes: radioactive decay, expanding gas, simple chemical reactions, radiation from a centre, photons sent through a half-silvered mirror, dice throwing, and an experiment with a ball balancing on a jet of water. Furthermore, the *law of statistical time asymmetry* finds support in a meta-symmetric time asymmetry described in Skoruppa (2022e). This long-ranging applicability of the law opens the possibility that it can describe the time asymmetry of the universe as a whole and thereby over-bridge the gap between time asymmetries at different levels of magnitude. This possibility will be investigated in a simple model in the following sections.

3 Assumptions behind the cosmological model

Before a model of the time asymmetric evolution of the universe can be constructed, some essential features must be established. Since the *law of general statistical time asymmetry* is aimed at being applied to the universe as a whole, the evolution of universe is assumed to be correctly describable in terms of probability. This assumption finds solid support in the laws of physics.

Up to now, the most fundamental description of our universe can be found in the equations of quantum mechanics, which, when applied to measured states, describe indeterministic evolutions ruled by probabilities. These quantum probabilities are, as mentioned in Section 1, universally applicable only when predicting the result of a *subsequent* measurement on the basis of a known measurement – that is when they are assumed to be future-directed. However, the time asymmetry of quantum mechanics is among the phenomena that a cosmological foundation is searched for and, therefore, quantum mechanics will initially be assumed to be time symmetric when applied to measured states. Such a point of departure is in accordance with the equations of quantum mechanics, which are time symmetric with respect to their mathematical formulation.[12] Accordingly, in the model analysed in Sections 4-7, the time symmetric probabilistic laws are represented by time translation invariant conditional probabilities, which values remain the same even if the conditional and outcome states are exchanged.

The most probable evolution, according to such time symmetric quantum theory, is a universe that, most of the time, is in macroscopic states, corresponding to the largest amount of possible microscopic permutations. Since this means statistical equilibrium, the most probable evolution of the universe, according to a time symmetric application of quantum theory, is a universe that starts in a state of equilibrium and thereafter only for relatively short periods departs from equilibrium.[13] This implies that the universe ought to be in a more or less constant state of almost evenly distributed radiation, rather than the aggregations of matter and energy found in the world around us today.

The question arises, what additional properties or boundary conditions that are required to equip a lifeless and time symmetric equilibrium universe with the rich complexity that signifies our current universe. If we accept the assumption of many possible evolutions of the universe, the following condition would make the observed evolution of our universe plausible:[14] An evolution including the actual state of universe at

[12]The question, posed by Callender (2000), whether a true time reversal of the Schrödinger equation should involve complex conjugation is ignored in this connection. As Gell-Mann & Hartle (1994, p. 312) puts it: "However, a time-neutral formulation of quantum mechanics allows us to investigate to what extent the familiar final condition of indifference with respect to future states is mandated by our observations." For a more detailed analysis of statistical time asymmetry in quantum mechanics, see Skoruppa (2022b).

[13]Gell-Mann & Hartle (1994, p. 318) write: "Relaxation to equilibrium is a time-symmetric process in a universe with an underlying dynamics that is time reversal invariant. Without boundary conditions, a system out of equilibrium is just as likely to have evolved from a state of higher entropy as it is to evolve to a state of higher entropy."

[14]Gell-Mann & Hartle (1994) make similar assumptions.

Big Bang, or a state that is very similar to it, has a higher probability than any other alternative evolution built on the physical laws of our universe. That is to say, a universe such as ours is at one of its assumed end points in time probably extremely dense and rapidly expanding, with all energy close to perfectly evenly distributed. It could even be the case, that this state in some sense is a necessary condition for the existence of the universe as we know it.

This line of reasoning suggests that the only condition needed – in addition to the energy content, the physical constants and the time symmetric quantum theory – for the actual time asymmetry and complexity of our actual universe to appear, is the very special initial boundary condition. If these ingredients are provided, the thermodynamical time asymmetry, the quantum time asymmetry, the emergency of life and all the other time asymmetries (except the expansion of the universe which is included in the condition) are supposed to emerge.

The *law of general statistical time asymmetry*, presented in Section 2, is supposed to provide a general description of various time symmetries. Hence, according to the suggestion above, this law ought to be a result of the combination of the time symmetric probabilistic laws of the universe and a given initial state. The aim of this work is to investigate if this can be verified in a simple model of the universe.

Finally, in order to make the model easy to analyse, the evolution between two states is assumed to be dependent on conditional probabilities, for which the only and sufficient condition is constituted by the nearest state in time. In other words, the evolution is assumed to be a *Markov chain*.

These desired features of a suitable model, which is intended to represent the time asymmetric evolution of universe, can be summarized as follows:

- the system is governed by probabilistic laws, i.e. time translation invariant non-zero probabilities;
- the system has a possible equilibrium state;
- the fundamental probabilistic laws are time symmetric;
- the time asymmetry is provided by the necessity of a non-equilibrium state at one of the temporal ends of the evolution; and
- the evolution is a Markov chain.

In Section 4, a simple model, build on these principles, is presented.

4 A time symmetric Ehrenfest model of the universe

In this and the following three sections, the consequences of different conditions are studied in a simple model of the universe to find the basis for time asymmetry of physical processes. First, the model is constructed with time symmetric laws, and then, in the next section, the influence of a boundary condition for the evolution of the system is examined.

In order to keep the mathematical calculations as lucid as possible, Ehrenfest's (1907) urn model is chosen because its simplicity. It also fills the requirements needed for the purpose of this analysis according to the criteria in the end of Section 3.

The model consists of two urns, A and B, containing m balls, which can be in 2^m different "microscopic" states, $S_j[t] \in \{S_0[t], S_1[t], S_2[t], \ldots, S_{2^m}[t]\}$, with regard to which urn each ball is situated in at the point in time t. Moreover, the balls can be in $m+1$ different "macroscopic" states with regard to how many balls are contained in each urn. With a period of one unit of time, the result of a fair lottery decides whether any ball should be moved and, in such case, which ball is moved from one urn to the other.

The process is, according to the assumption of a fair lottery above, a *homogeneous Markov process*. The Markov property implies that for any $t_1 < t_2 < \ldots < t_k < t_{k+1} < \ldots < t_n$ or $t_1 > t_2 > \ldots > t_k > t_{k+1} > \ldots > t_n$, where the succession of states $S_1[t_1], S_2[t_2], \ldots, S_k[t_k], \ldots, S_n[t_n]$ is an arbitrary evolution in the time interval $[t_1, t_n]$,

$$p(S_{k+1}[t_{k+1}] \wedge S_{k+2}[t_{k+2}] \wedge \ldots \wedge S_n[t_n] \,|\, S_1[t_1] \wedge \ldots \wedge S_k[t_k]) = \\ = p(S_{k+1}[t_{k+1}] \wedge S_{k+2}[t_{k+2}] \wedge \ldots \wedge S_n[t_n] \,|\, S_k[t_k]) \ ,$$

The time evolution of the unconditional probabilities can, in terms of future-directed probability as defined in the Section 2, be expressed as

$$p(S_i[t_b]) = \sum_{j=1}^{2^m} p(S_i[t_b]|S_j[t_a]) \, p(S_j[t_a]) \ ,$$

where $t_a < t_b$. This can be rewritten in more compact form as

$$\boldsymbol{p}(t_b) = P_m(t_b|t_a) \, \boldsymbol{p}(t_a) \ ,$$

where the probability vectors $\boldsymbol{p}(t_a)$ and $\boldsymbol{p}(t_b)$ are representations of the probability distributions

$$p(S_1[t_a]), p(S_2[t_a]), p(S_3[t_a]), \ldots, p(S_{2^m}[t_a]) \text{ and} \\ p(S_1[t_b]), p(S_2[t_b]), p(S_3[t_b]), \ldots, p(S_{2^m}[t_b]),$$

respectively, and $P_m(t_b|t_a)$ is the stochastic future-directed transition matrix of an Ehrenfest model with m balls mapping the probability vector $\boldsymbol{p}(t_a)$ into $\boldsymbol{p}(t_b)$.

Next, the evolution of the model is assumed to be time symmetric with no given boundary conditions. For this purpose, the formulation of *probabilistic time reversal invariance*, presented in two recent articles,[15] is applied:

$$p(S_j[t_a + \Delta t]|S_i[t_a]) = p(S_j[t_a - \Delta t]|S_i[t_a]) \ ,$$

where $\Delta t = t_b - t_a$, and where t_a and t_b are two arbitrary points in time. This implies

$$\begin{aligned} & P(t_b|t_a) = P(2t_a - t_b|t_a) \\ \Rightarrow & \{\text{time translation invariance}\} \\ \Rightarrow & P(t_b|t_a) = P(t_a|t_b) \ , \end{aligned} \tag{1}$$

which demonstrates that the matrix description is time symmetric.

Since a fair lottery rules the transitions, the following equality holds for any pair of states S_i and S_j at any pair of points in time t_a and t_b, where $t_a < t_b$:[16]

$$p(S_i[t_b]|S_j[t_a]) = p(S_j[t_b]|S_i[t_a]) \ , \tag{2}$$

and as a consequence of the time symmetry (1):

$$\begin{aligned} & \left.\begin{array}{l} \{(1)\} \Rightarrow p(S_i[t_b]|S_j[t_a]) = p(S_i[t_a]|S_j[t_b]) \\ \{(1)\} \Rightarrow p(S_j[t_b]|S_i[t_a]) = p(S_j[t_a]|S_i[t_b]) \end{array}\right\} \Rightarrow \{(2)\} \Rightarrow \\ & \Rightarrow p(S_i[t_a]|S_j[t_b]) = p(S_j[t_a]|S_i[t_b]) \ . \end{aligned} \tag{3}$$

which is the time reversal of equation (2).[17] As a result of the time reversal, the symmetry (3) is an equality between two past-directed probabilities, while equation (2) is an equality between two future-directed probabilities.

[15] This symmetry is analysed in Holster (2003) and Skoruppa (2022e), where it is shown to be a necessary and sufficient criterion for probabilistic time reversal invariance. Holster has coined it *the correct criterion for time reversal invariance.*

[16] This symmetry is conventionally named *time reversal invariance*, but Holster (2003) and Skoruppa (2022e) point out that it is neither a necessary nor a sufficient criterion for time symmetry. It is therefore named *the orthodox criterion for reversal symmetry* by Holster (2003) and *criterion for future-directed process reversal invariance* by Skoruppa (2022e).

[17] Symmetry (3) is analysed by Skoruppa (2022e), where it is named *criterion for past-directed process reversal invariance.* It is the time reversed counterpart of symmetry represented in equation (2).

Moreover, for any pair of states S_i and S_j at any pair of points in time t_a and t_b:

$$\begin{aligned}\{(1)\} &\Rightarrow p(S_j[t_b]|S_i[t_a]) = p(S_j[t_a]|S_i[t_b]) \Rightarrow \{(2)\}\\ &\Rightarrow p(S_i[t_b]|S_j[t_a]) = p(S_j[t_a]|S_i[t_b]) \Rightarrow \{\text{definition}\}\\ &\Rightarrow \frac{p(S_i[t_b] \wedge S_j[t_a])}{p(S_j[t_a])} = \frac{p(S_j[t_a] \wedge S_i[t_b])}{p(S_i[t_b])}\\ &\Rightarrow p(S_j[t_a]) = p(S_i[t_b]) \ .\end{aligned}$$

Obviously, the only possible state vector satisfying this equality, for any pair of states S_i and S_j at any pair of points in time t_a and t_b, is a steady-state vector

$$\boldsymbol{p}_s(t) = \begin{bmatrix} p(S_1[t]) \\ p(S_2[t]) \\ \vdots \\ p(S_{2^m}[t]) \end{bmatrix} = 2^{-m} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \ . \tag{4}$$

Altogether this means, for the stochastic model described above, that time symmetry implies a steady state *ensemble*, or in other words *statistical equilibrium*. Time translation invariant time symmetry thus brings about a constant distribution of probabilities, as described by equation (4), with equal probability for every possible micro-state with regard to the positions of the balls.

The time symmetric evolution is assumed only to be constrained by two conditions: the given transition probabilities and the condition of time symmetry. This allows the actual process to be chosen randomly between all possible trajectories of the system, with the probability of each trajectory given by these conditions. As a part of the model of a universe, these two conditions represent the physical *laws*.

It is noteworthy that the outcome trajectory of this model universe does not have to be time symmetric with respect to the initial and final states, even though the laws describing the universe are time symmetric. This would be true even if the evolution was not constrained by any given event, since a randomly chosen trajectory has a very small probability of actually being time symmetric, even if only the two conditions mentioned in the previous paragraph were given. For example, there are probably more balls in one urn than in the other in the initial state (if the total number is greater than 2), and the final state could be the result of a fluctuation in the opposite direction.

Still, the most *probable* evolutions of the model universe start in a state close to an even distribution and remain in a similar state most of the time, since a majority of the equally probable states represents such distributions. In other words, the time symmetric model universe probably has its initial and final states close to precise equilibrium and, in between, fluctuates around the equilibrium state. However, if the lifetime of this model universe is very long, some of the fluctuations will reach far from equilibrium.

5 Evolution of the model when constrained by a boundary condition

The next step in the analysis of the urn model is to constrain the time symmetric stochastic evolution, described in the last section, by a given boundary condition. Thereafter, this modified model will be analysed with respect to whether it develops in accordance with *law of general statistical time asymmetry*.

First it is investigated how a future-directed probability for the transition between two arbitrary states S_i och S_j,

$$p(S_j[t + \Delta t]|S_i[t])_{S[t_0]} \ ,$$

is influenced by a given initial boundary condition $S[t_0]$ (denoted by a subscript), where $t > t_0$, $\Delta t > 0$. Assume that $S[t_0]$ is compatible with $S_i[t]$, i.e. $p(S[t_0] \wedge S_i[t]) \neq 0$. Thus,

$$\begin{aligned} & p(S_j[t + \Delta t]|S_i[t])_{S[t_0]} = \\ & = p(S_j[t + \Delta t]|S[t_0] \wedge S_i[t]) = \{t_0 < t < t + \Delta t; \text{Markov process}\} \\ & = p(S_j[t + \Delta t]|S_i[t]) \ . \end{aligned} \tag{5}$$

This implies that the only constraint that the initial boundary condition $S[t_0]$ puts on the future-directed probabilities is that the conditional state is compatible with $S[t_0]$. Besides that, the future-directed probability is independent of a given initial boundary condition. Hence, the future-directed probabilities keep their time translation invariance (i.e. in comparison with when the model is time symmetric with no given boundary conditions) also when the process is constrained by an initial boundary condition, given that the probabilities are still definable.

Next, it will be shown how an arbitrary *past-directed probability*, $p(S_i[t]|S_j[t + \Delta t])$, is influenced by a given initial boundary condition, $S[t_0]$, where $t > t_0$ and where the given condition is denoted by a subscript. Again, $S_j[t_0]$ is assumed to be compatible with the arbitrary following state $S_j[t+\Delta t]$, i.e. $p(S[t_0] \wedge S_j[t+\Delta t]) \neq 0$. This constrained past-directed probability can be formulated in terms of future-directed probabilities by repeatedly applying the definition of conditional probability:

$$
\begin{aligned}
& p(S_i[t]|S_j[t+\Delta t])_{S[t_0]} \\
&= p(S_i[t]|S[t_0] \wedge S_j[t+\Delta t]) \\
&= \{p(S[t_0] \wedge S_j[t+\Delta t]) \neq 0\} \\
&= \frac{p(S[t_0] \wedge S_i[t] \wedge S_j[t+\Delta t])}{p(S[t_0] \wedge S_j[t+\Delta t])} \\
&= \frac{p(S[t_0] \wedge S_i[t]) \cdot p(S_j[t+\Delta t]|S[t_0] \wedge S_i[t])}{p(S[t_0] \wedge S_j[t+\Delta t])} \\
&= \frac{p(S[t_0]) \cdot p(S_i[t]|S[t_0]) \cdot p(S_j[t+\Delta t]|S[t_0] \wedge S_i[t])}{p(S[t_0]) \cdot p(S_j[t+\Delta t]|S[t_0])} \\
&= \frac{p(S_i[t]|S[t_0]) \cdot p(S_j[t+\Delta t]|S[t_0] \wedge S_i[t])}{p(S_j[t+\Delta t]|S[t_0])} \\
&= \{t_0 < t < t+\Delta t; \text{Markov process}\} \\
&= \frac{p(S_i[t]|S[t_0])}{p(S_j[t+\Delta t]|S[t_0])} \cdot p(S_j[t+\Delta t]|S_i[t]) \, . \qquad (6)
\end{aligned}
$$

It has already been concluded in connection with equation (5) that the future-directed probability $p(S_j[t+\Delta t]|S_i[t])$ is time translation invariant. This means that the arbitrary past-directed probability $p(S_i[t]|S_j[t+\Delta t])_{S[t_0]}$ is time translation invariant if and only if the quotient in expression (6) is time translation invariant. However, as will be exemplified for $m = 2$ (i.e. in a model with two balls) in Section 6, this quotient is usually dependent on t.

With a corresponding procedure as in the derivation of equation (5), it can be shown that a given final boundary condition, $S[t_f]$, where $t_0 < t < t + \Delta t < t_f$, does not influence the time translation invariance of the past-directed probabilities (again, in comparison with when the model is time symmetric with no given boundary conditions), since

$$
\begin{aligned}
& p(S_i[t]|S_j[t+\Delta t])_{S[t_f]} = \{p(S_j[t+\Delta t] \wedge S[t_f]) \neq 0\} \\
&= p(S_i[t]|S_j[t+\Delta t] \wedge S[t_f]) = \{0 \leq t \leq t+\Delta t; \text{Markov process}\} \\
&= p(S_i[t]|S_j[t+\Delta t]) \, , \qquad (7)
\end{aligned}
$$

given that $p(S_j[t+\Delta t] \wedge S[t_f]) \neq 0$.

On the other hand, corresponding to the derivation of equation (6), a given final boundary condition makes the *future-directed* probability dependent on t, since

$$
p(S_j[t+\Delta t]|S_i[t])_{S[t_f]} = \ldots = \frac{p(S[t_f]|S_j[t+\Delta t])}{p(S[t_f]|S_i[t])} \cdot p(S_i[t]|S_j[t+\Delta t]) \, , \qquad (8)
$$

which cannot be expected to be time translation invariant of the same reasons as equation (6).

It is now clear that the evolution in the urn model is in accordance with the *law of general statistical time asymmetry*, if there is a given *initial* boundary condition in an otherwise time symmetric stochastic evolution. This follows from the fact that the future-directed transition probabilities but not the past-directed probabilities are time translation invariant under these circumstances, according to equations (5) and (6).

By a corresponding line of reasoning it can also be shown, according to equations (7) and (8), that a time symmetric urn model, in combination with a given final boundary condition, is not in accordance with the *law of general statistical time asymmetry.* Instead, such conditions implies an evolution of the system in accordance with time translation invariant past-directed transition probabilities, whereas the future-directed probabilities are generally not time translation invariant.

6 The two-ball model universe

In order to clarify the analysis in Section 5, it is suitable to study the actual evolution of a specified Ehrenfest model. The simplest model of this kind that both can demonstrate time-directed behaviour and has a possible equilibrium state consists of two balls, 1 and 2, that can be in either of the two urns, A and B. This system can be in four states, which in the following are denoted by binary numbers: 00, 01, 10 and 11. The first position is 0 if ball number 1 is in urn A and 1 if it is in urn B, and the second position represents the state of ball number 2 in a corresponding way. The probability of the state to be preserved during a positive time interval is assumed to be non-zero and the time unit is defined as the time interval when this probability is $\frac{1}{2}$.

Since the two balls have equal probability to shift urn, a simple case of transition probabilities that correspond to the assumptions above can be described with two equal matrices in accordance with the time symmetric equation (1):

$$P_2(t+1|t) = P_2(t|t+1) = \frac{1}{4}\begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix},$$

where the subscript in P_2 denotes that it is a two-ball model, and where

$$P_2(t+1|t) = \\ = \begin{bmatrix} p(00[t{+}1]|00[t]) & p(00[t{+}1]|01[t]) & p(00[t{+}1]|10[t]) & p(00[t{+}1]|11[t]) \\ p(01[t{+}1]|00[t]) & p(01[t{+}1]|01[t]) & p(01[t{+}1]|10[t]) & p(01[t{+}1]|11[t]) \\ p(10[t{+}1]|00[t]) & p(10[t{+}1]|01[t]) & p(10[t{+}1]|10[t]) & p(10[t{+}1]|11[t]) \\ p(11[t{+}1]|00[t]) & p(11[t{+}1]|01[t]) & p(11[t{+}1]|10[t]) & p(11[t{+}1]|11[t]) \end{bmatrix}$$

and

$$P_2(t|t+1) = \\ = \begin{bmatrix} p(00[t]|00[t{+}1]) & p(00[t]|01[t{+}1]) & p(00[t]|10[t{+}1]) & p(00[t]|11[t{+}1]) \\ p(01[t]|00[t{+}1]) & p(01[t]|01[t{+}1]) & p(01[t]|10[t{+}1]) & p(01[t]|11[t{+}1]) \\ p(10[t]|00[t{+}1]) & p(10[t]|01[t{+}1]) & p(10[t]|10[t{+}1]) & p(10[t]|11[t{+}1]) \\ p(11[t]|00[t{+}1]) & p(11[t]|01[t{+}1]) & p(11[t]|10[t{+}1]) & p(11[t]|11[t{+}1]) \end{bmatrix}$$

According to equation (4), this implies that this time symmetric evolution is correctly described by a steady-state vector

$$\begin{bmatrix} p(00[t]) \\ p(01[t]) \\ p(10[t]) \\ p(11[t]) \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} .$$

Assume that $n \in \mathbf{N}$.[18] Then, a general expression for the transition probabilities of the time symmetric evolution is (with I_4 denoting the identity matrix)

$$\begin{aligned} P_2(t+n|t) &= P_2(t|t+n) \\ &= \begin{cases} \dfrac{1}{4^n} \begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix}^n, \text{ if } n > 0. \\ I_4, \text{ if } n = 0. \end{cases} \\ &= \begin{cases} \dfrac{1}{2^{n+2}} \begin{bmatrix} 2^n+2 & 2^n & 2^n & 2^n-2 \\ 2^n & 2^n+2 & 2^n-2 & 2^n \\ 2^n & 2^n-2 & 2^n+2 & 2^n \\ 2^n-2 & 2^n & 2^n & 2^n+2 \end{bmatrix}, \text{ if } n > 0. \\ I_4, \text{ if } n = 0. \end{cases} \end{aligned} \tag{9}$$

as can be shown by induction. Somewhat surprisingly, in this general two-ball matrix, half of the entries are independent of n. However, it

[18]In the following, the number of transitions, n, instead of Δt, is denoting time difference in the probabilities, in order to make the expressions clearer.

can easily be shown that this is the case only when the probability for a maintained state after one unit of time is $\frac{1}{2}$.

Now, assume that this two-ball universe is constrained by the boundary condition 00[0], which denotes that the system is in the state 00 at $t = 0$ and, furthermore, assume that we want to describe the evolution of the system for $t \geq 0$. Then, according to equation (5) and with the subscript denoting the given boundary condition, the future-directed transition probabilities after $n \in \mathbf{N}^+$ transistions are unaffected by the given initial state:

$$P_2(t+n|t)_{00[0]} = P_2(t+n|t) = \{(9)\}$$
$$= \begin{cases} \frac{1}{2^{n+2}} \begin{bmatrix} 2^n+2 & 2^n & 2^n & 2^n-2 \\ 2^n & 2^n+2 & 2^n-2 & 2^n \\ 2^n & 2^n-2 & 2^n+2 & 2^n \\ 2^n-2 & 2^n & 2^n & 2^n+2 \end{bmatrix}, \text{ if } n > 0. \\ I_4, \text{ if } n = 0. \end{cases} \tag{10}$$

Consequently, the future-directed probabilities are independent of t and thereby time translation invariant.

On the other hand, when constrained by an initial state 00[0], the *past-directed* probabilities are modified according to equation (6) in combination with the future-directed probabilities from equation (10):

$$P_2(t|t+n)_{00[0]} = \begin{bmatrix} \frac{(2^t+2)\cdot(2^n+2)}{4\cdot(2^{t+n}+2)} & \frac{(2^t+2)\cdot 2^n}{4\cdot 2^{t+n}} & \frac{(2^t+2)\cdot 2^n}{4\cdot 2^{t+n}} & \frac{(2^t+2)\cdot(2^n-2)}{4\cdot(2^{t+n}-2)} \\ \frac{2^{t+n}}{4\cdot(2^{t+n}+2)} & \frac{2^t\cdot(2^n+2)}{4\cdot 2^{t+n}} & \frac{2^t\cdot(2^n-2)}{4\cdot 2^{t+n}} & \frac{2^{t+n}}{4\cdot(2^{t+n}-2)} \\ \frac{2^{t+n}}{4\cdot(2^{t+n}+2)} & \frac{2^t\cdot(2^n-2)}{4\cdot 2^{t+n}} & \frac{2^t\cdot(2^n+2)}{4\cdot 2^{t+n}} & \frac{2^{t+n}}{4\cdot(2^{t+n}-2)} \\ \frac{(2^t-2)\cdot(2^n-2)}{4\cdot(2^{t+n}+2)} & \frac{(2^t-2)\cdot 2^n}{4\cdot 2^{t+n}} & \frac{(2^t-2)\cdot 2^n}{4\cdot 2^{t+n}} & \frac{(2^t-2)\cdot(2^n+2)}{4\cdot(2^{t+n}-2)} \end{bmatrix}. \tag{11}$$

It is clear that the past-directed probabilities are dependent on t and thereby are *not* time translation invariant. Hence, the evolution in this two-ball universe with a given initial state demonstrates time translation invariant future-directed probabilities, while the past-directed probabilities vary with time. Consequently, the evolution of the two-ball model, constrained by the 00 initial boundary condition, is in accordance with the *law of general statistical time asymmetry.*

7 Expected evolutions of the model

The two-ball model can be used to illustrate the statistically *expected* evolution of the urn model. In the case of the time symmetric version described in Section 4, the steady state vector (4) implies that the the expected evolution simply is a constant statistical equilibrium. In the case of a *given initial state*, 00[0], the expected number of balls in urn B at the point of time $t \in \mathbf{N}^+$, $N_0[t]$, is given by

$$\begin{aligned} N_0[t] &= 0 \cdot p(00[t]|00[0])_{00[0]} + 1 \cdot p(01[t]|00[0])_{00[0]} \\ &\qquad + 1 \cdot p(10[t]|00[0])_{00[0]} + 2 \cdot p(11[t]|00[0])_{00[0]} = \{(10)\} \\ &= 2 \cdot \frac{2^t}{2^{t+2}} + 2 \cdot \frac{2^t - 2}{2^{t+2}} \\ &= 1 - 2^{-t} \, . \end{aligned} \tag{12}$$

.

The two-ball model can also illustrate the influence of *both* a given initial and a given final state. Let the initial and final state be 00[0] and $00[t_f]$, where $0 \le t \le t_f$. Then the expected number of balls in urn B at the point of time $t \in \mathbf{N}^+$, $N_{[0,t_f]}(t)$, is

$$\begin{aligned} N_{[0,t_f]}(t) &= 0 \cdot p(00[t]|00[0] \wedge 00[t_f]) + 1 \cdot p(01[t]|00[0] \wedge 00[t_f]) \\ &\qquad + 1 \cdot p(10[t]|00[0] \wedge 00[t_f]) + 2 \cdot p(11[t]|00[0] \wedge 00[t_f]) \, . \end{aligned}$$

This can be expressed in terms of past-directed probabilities according to the matrix (11):

$$\begin{aligned} N_{[0,t_f]}(t) &= 1 \cdot p(01[t]|00[0] \wedge 00[t_f]) \\ &\qquad + 1 \cdot p(10[t]|00[0] \wedge 00[t_f]) + 2 \cdot p(11[t]|00[0] \wedge 00[t_f]) \\ &= p(01[t]|00[t_f])_{00[0]} + p(10[t]|00[t_f])_{00[0]} + 2 \cdot p(11[t]|00[t_f])_{00[0]} \\ &= \{(11)\} \\ &= 2 \cdot \frac{2^{t_f}}{4 \cdot (2^{t_f} + 2)} + 2 \cdot \frac{(2^t - 2) \cdot (2^{t_f - t} - 2)}{4 \cdot (2^{t_f} + 2)} \\ &= 1 - \frac{2^t + 2^{t_f - t}}{2^{t_f} + 2} \, . \end{aligned} \tag{13}$$

This implies that the initial expected evolution, given both an initial and a final state according to (13), asymptotically approaches the expected evolution, given an initial state according to (12), as the time interval between the initial and final state increases:

$$\lim_{t_f \to \infty} 1 - \frac{2^t + 2^{t_f - t}}{2^{t_f} + 2} = \lim_{t_f \to \infty} 1 - \frac{2^{t - t_f} + 2^{-t}}{1 + 2^{1 - t_f}} = 1 - 2^{-t} \, .$$

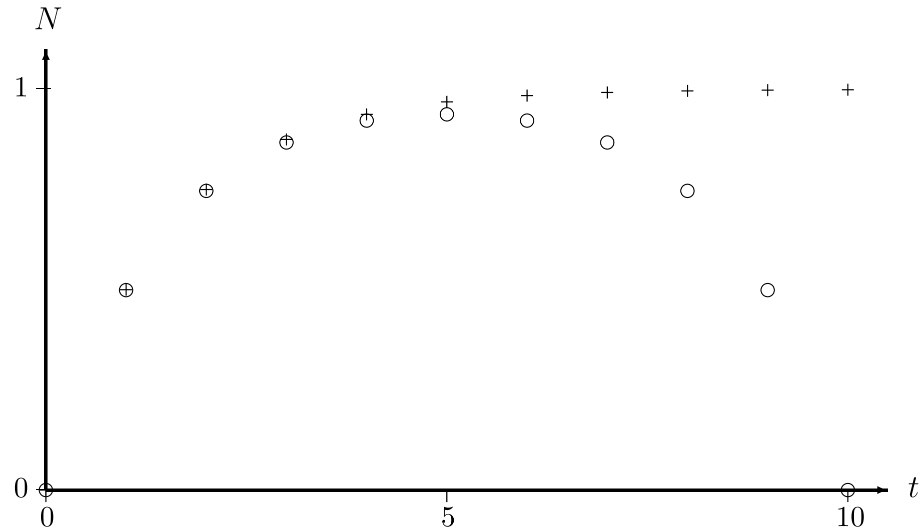

Figure 1: Expected evolution in a two-ball Ehrenfest urn-model constrained by an initial state (crosses) and by both an initial and a final state (circles).

Consequently, at the initial stage of the two-ball model it is difficult to empirically distinguish between the evolution governed by time-symmetric probabilistic "laws" solely constrained by a given initial state from the corresponding evolution constrained by a given final state in addition to the same initial state. This can be illustrated through the study of both types of evolutions, given a specified time interval between the initial and the final state. In Figure 1 both these evolutions are depicted in the case of a two-ball "universe", where the time span between the two given boundary condition is ten time units, i.e. $t_f = 10$:

$$\text{crosses: } N = 1 - 2^{-t} \,,$$
$$\text{circles: } N = 1 - \frac{2^t - 2^{10-t}}{1026} \,.$$

The difference between the two evolutions at $t = 1$ is less than 0.2 percent.

8 Discussion

The results from the previous sections imply that an Ehrenfest urn model, fundamentally governed by time symmetric transition probabilities, evolves asymmetrically in time when constrained by a given initial state. This time asymmetry is evident through the time translation invariance of the future-directed probabilities, while the past-directed probabilities lack this kind of invariance. This is exactly what the *law of general statistical time asymmetry* prescribes.

This analysis of the Ehrenfest model indicates that the only thing needed to provide statistical time asymmetry, is that the initial state of the evolution is given by some principle that is independent of the fundamental probabilistic laws. If the same principle is applicable to our universe, the very special state of Big Bang constitutes the foundation for physical statistical time asymmetry in the form of the *law of general statistical time asymmetry*. As was shown in Section 2, this law has the potential to – in combination with time symmetric physical laws – describe all time asymmetric physical processes except the evolution of the universe as a whole. Hence, if the structure of the initial state of our universe is given by some unknown principle or law of nature, the entire time asymmetry of our universe could be explained.

References

Bohm, A. R., Gadella, M., & Kielanowski, P. (2011). Time asymmetric quantum mechanics. *Sigma*, 7, 086.

Boltzmann, L. (1897). Zu Hrn. Zermelos Abhandlung "Über die mechanische Erklärung irreversibler Vorgänge". *Annalen der Physik*, 60, 392–398.

Born, M. (1948). *Natural Philosophy of Cause and Chance.* Oxford: Clarendon Press.

Callender, C. (2000). Is time 'handed' in a quantum world? *Proceedings of the Aristotelian Society*, 100, 247–269.

Carroll, S. M. & Chen, J. (2004). Spontaneous inflation and the origin of the arrow of time. Available at http://arxiv.org/abs/hep-th/0410270v1.

Cocke, W. J. (1967). Statistical time symmetry and two-time boundary conditions in physics and cosmology. *Physical Review*, 160, 1165–1170.

Cramer, J. G. (2000). The plane of the present and the new transactional paradigm of time. In R. Durie (Ed.), *Time & the Instant* (pp. 177–189). Manchester: Clinamen Press.

Denbigh, K. G. (1989). Note on entropy, disorder and disorganization. *The British Journal for the Philosophy of Science*, 40, 323–332.

Earman, J. (1974). An attempt to add a little direction to "the problem of the direction of time". *Philosophy of Science*, 41, 15–47.

Earman, J. (2006). "The past hypothesis": Not even false. *Studies in History and Philosophy of Modern Physics*, 37, 399–430.

Ehrenfest, P. & Ehrenfest, T. (1907). Über zwei bekannte Einwände gegen das Boltzmannsche H-theorem. *Physikalische Zeitschrift*, 8, 311–314.

Fock, V. (1948). On the interpretation of the wave function directed towards the past. *Doklady Akademii SSSR*, 60, 1157–1159.

Gell-Mann, M. & Hartle, J. B. (1994). Time symmetry and asymmetry in quantum mechanics and quantum cosmology. In J. J. Halliwell, J. Pérez-Mercader, & W. H. Zurek (Eds.), *Physical Origins of Time Asymmetry* (pp. 311–345). Cambridge: Cambridge University Press.

Gold, T. (1962). The arrow of time. *American Journal of Physics*, 30, 403–410.

Holster, A. (2003). The criterion for time symmetry of probabilistic theories and the reversibility of quantum mechanics. *New Journal of Physics*, 5, 130.1–130.28.

Landau, L. D. & Lifshitz, E. M. (1958). *Statistical Physics.* London: Pergamon Press.

Mehlberg, H. (1961). Physical laws and time's arrow. In H. Feigl & G. Maxwell (Eds.), *Current Issues in the Philosophy of Science* (pp. 105–138). New York: Holt, Rinehart and Winston.

Penrose, R. (1989). *The Emperor's New Mind.* Oxford: Oxford University Press.

Reichenbach, H. (1956). *The Direction of Time.* Berkley: University of California Press.

Skoruppa, B. (2022a). Derivation of rate equations from time-directed probabilities of individual particles. Appendix IV.

Skoruppa, B. (2022b). Probabilistic time asymmetry in quantum mechanical processes. Appendix VI.

Skoruppa, B. (2022c). A proposed law for statistical time asymmetry. Appendix II.

Skoruppa, B. (2022d). Time asymmetry in macroscopic stochastic processes. Appendix V.

Skoruppa, B. (2022e). Time reversal invariance and asymmetry in probability based laws. Appendix I.

Sober, E. (1993). Temporally oriented laws. *Synthese*, 94, 171–189.

Uffink, J. (2007). Compendium of the foundations of classical statistical physics. In I. J. Butterfield & J. Earman (Eds.), *Philosophy of Physics* (pp. 923–1074). Amsterdam: Elsevier.

Watanabe, S. (1955). Symmetry of physical laws. Part III. Prediction and retrodiction. *Reviews of Modern Physics*, 27, 179–186.

Watanabe, S. (1966). Time and the probabilistic view of the world. In J. T. Fraser (Ed.), *The Voices of Time* (pp. 527–563). New York: George Braziller.