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Appendix I. Time reversal invariance and asymmetry in probability based laws

Abstract

To provide tools for describing phenomena that show asymmetry of time, four symmetries of time dependent conditional probabilities are identified. The symmetries are analysed with respect to their mathematical properties and relations, and their characteristic features are demonstrated in two models as well as in the statistical description of thermodynamic equilibrium and detailed balance. On the basis of these analyses two criteria are defined: a criterion for probabilistic time reversal invariance and a criterion for future-directed process reversal invariance. The significance of the four symmetries with regard to physics is described, whereby a meta-symmetric time asymmetry is revealed. This asymmetry manifests itself through the fact that the so called T-invariance, which is based on future-directed probabilities, has lawlike status in physics, while its time reversed counterpart does not.

1 Introduction

Several terms have been adopted to describe physical phenomena showing various kinds of differences between time direction towards the future and towards the past: time asymmetry, time anisotropy, arrow of time, temporal orientation, direction of time, irreversibility, time reversal invariance, and T-invariance. There is a lack of consensus on the interpretation of these concepts, and in some works authors have tried to clarify their meaning and interrelations.[1] The need for a solid conceptual basis in order to solve the foundational problems of thermodynamics and statistical mechanics implies that clearer definitions are desirable.[2] To make a contribution to such clarification, the present work concentrates on finding ways to describe symmetries in terms of time dependent conditional probabilities.

To some extent the present work corresponds to the analysis of Holster (1990, 2003a), which is founded on ideas originating from Watanabe (1955, 1965). The present article, like the work of Holster, aims to elucidate the distinction between the *time asymmetry* and *time reversal*

[1]See, for example, Bunge (1972), Davies (1974), Earman (1974), and Zeh (1992).
[2]Uffink (2001, 2007) describes these foundational problems.

invariance of stochastic processes and to propose a definition of time reversal invariance applicable to both deterministic and stochastic processes. However, in many respects the present analysis is carried out from other starting points and, consequently, somewhat different conclusions are drawn.

The most important differences in relation to Holster (2003a) are:

- a different derivation of the probabilistic symmetries, clarifying their interrelations and meanings;
- an analysis of what is here called *Symmetry IV*, not mentioned by Holster;
- a more detailed analysis of the correspondence between deterministic time reversal invariance and the probabilistic symmetries;
- a more thorough analysis of how the probabilistic symmetries relates to thermodynamic equilibrium; and
- the establishment of a time asymmetry according to which (terms introduced here) *Symmetry III*, but not *Symmetry IV*, has significance in physics.

The wealth of details in the present investigation of the characteristics of probabilistic symmetries is motivated by the aforementioned lack of consensus concerning what time reversal actually means in the realm of statistical physics. The aim is to provide a basis for a unified definition of statistical time symmetry.

The outline of the paper is as follows. In Sections 2, 3, and the beginning of Section 4, basic concepts, with regard to the analysis of time symmetries, are defined. Four unique probabilistic symmetries are formulated at the end of Section 4, and these symmetries are analysed and exemplified in Sections 5 – 8. In Section 9, the validity of the symmetries under the condition of thermodynamic equilibrium is investigated. Two of the probabilistic symmetries are given an explicit physical definition in Section 10, after which the physical significance of all four probabilistic symmetries is described in Section 11. Finally, the conclusions of the present article are summarized in Section 12.

2 Some basic definitions

To clear the way for the following analyses, some concepts must be defined. Since symmetry is the primary issue of the present work, it is important to clarify what is meant by mirroring something with respect to past and future, or in other words, what *time reversal* means.

Definition 1 (time reversal)
Time reversal denotes the application of the transformation $T : t \to -t$ to a law, probability statement, process, state, or a relation between such entities when described in mathematical terms.

Time reversal implies, for example, $v \to -v$ in Newton mechanics, $\{q, p\} \to \{q, -p\}$ in Hamilton mechanics, and $\{\boldsymbol{E}(\boldsymbol{r}), \boldsymbol{B}(\boldsymbol{r})\} \to \{\boldsymbol{E}(\boldsymbol{r}), -\boldsymbol{B}(\boldsymbol{r})\}$ in electrodynamics.[3]

Now another essential concept can be defined.

Definition 2 (time symmetry)
Time symmetry denotes that a probability statement, process, or a relation between such entities is not altered under time reversal.

Consequently, *time asymmetry* means that a mathematical description of a law, probability statement, process, or a relation between such entities is altered under time reversal. Therefore, the laws, statements, processes, and relations, rather than their mathematical descriptions, will be defined as *time symmetric* or *time asymmetric*. Furthermore, the concept *time direction* is used when one of the entities mentioned exhibits time asymmetry, whereby the order of its constituents define a direction with respect to past and future.

Time reversal invariance, a concept closely related to *time symmetry*, meets, at least when applied to classical mechanics, some degree of consensus among physicists.[4]

Definition 3 (time reversal invariance)
A law L is time reversal invariant, if for all t, [the physical evolution $z(t)$ satisfies L] $\Leftrightarrow$ [the physical evolution $z^T(-t)$ satisfies L], where $z^T = Tz$ is the time reversed state of z by the transformation $T : t \to -t$.

In this definition, as well as in the rest of the presentation, physical laws will be assumed to be time *translation* invariant, i.e. they are invariant with respect to the choice of origin of the time coordinate.

Fundamental to the analysis of physical time asymmetry is the distinction between processes that can be correctly described by laws and symmetries, prescribing only *one possible trajectory* between a given state and preceding and following states, and processes that can be correctly described by laws and symmetries *based on probabilities*. In the

[3]This originates from Zeh (1992).

[4]This definition is based on the “formulation” made by Zeh (1992), which is adapted to suit the context of the present article.

following account, the concept *deterministic* represents processes of the former kind, as well as laws and symmetries describing such processes. The concept *probabilistic* represents processes of the latter kind, as well as the laws and symmetries belonging to them.

3 Definitions of deterministic time reversal invariance

In previous works, other authors have concluded that an implication between two states can correctly be replaced by an equivalence, if the system follows non-probabilistic laws of physics, e.g. in the branches of physics that do not describe processes in terms of probabilities and statistics.[5] Hence, there is a reason to formulate the following postulate.

> **Postulate 1** If a process is correctly described for the states A and B at arbitrary moments t_A and t_B, respectively, by deterministic physical laws, the following implication holds:
>
> $$(A[t_A] \Rightarrow B[t_B]) \Rightarrow (A[t_A] \Leftrightarrow B[t_B]) \ .$$

The aim of this article is to make an analysis of probabilistic time symmetry. Since conditioned probability is formulated as a relation between two states, Definition 3 has to be reformulated in order to enable comparison with related definitions for probabilistic laws. This can be done by replacing $z(t)$ in Definition 3 with an evolution between two arbitrary states A and B.

Therefore, let $z(t) = (A[t] \Leftrightarrow B[t + \Delta t])$, where $A[t]$ and $B[t + \Delta t]$ denote that the system is in states A and B at t and $t + \Delta t$, respectively (Δt, here as well as in the rest of the article, is a positive time interval). It is noteworthy that $z(t)$, as a consequence, is a function that can assume either the value *true* or *false*. The transformation $T : t \rightarrow -t$ will now be applied to $z(t) = (A[t] \Leftrightarrow B[t + \Delta t])$ (with a superscript '^{T}' representing a time reversed state according to Definition 3):

$$\begin{aligned}
& T(A[t] \Leftrightarrow B[t + \Delta t]) \\
\Leftrightarrow\ & (A^T[-t] \Leftrightarrow B^T[-t - \Delta t]) \\
\Leftrightarrow\ & \{\text{time translation invariance}\} \\
\Leftrightarrow\ & (A^T[t + \Delta t] \Leftrightarrow B^T[t]) \ .
\end{aligned}$$

[5] Sober (1993, p. 171): "Non-probabilistic laws that describe a conditional relationship between earlier and later are not temporally oriented. A law of the form 'If the system is in state E at an earlier time, then it will be in state L at a later time' supports inferences in both directions." The only time-directed processes that can be considered as deterministic are to be found in thermodynamics, and if so, they would be in conflict with the postulate. However, according to statistical mechanics they are basically probabilistic (Liu, 1993) and therefore do not provide any counterexample.

This time reversal makes the following reformulation of Definition 3 possible:

Definition 4 (deterministic time reversal invariance). A deterministic physical law L_d is time reversal invariant if, for arbitrary states A and B and for all t and Δt,

$$(A[t] \Leftrightarrow B[t + \Delta t]) \Leftrightarrow (A^T[t + \Delta t] \Leftrightarrow B^T[t])$$

is true according to L_d.

The distinction between a general concept of time reversal invariance and deterministic time reversal invariance is essential in order to understand why there are such difficulties generalizing the concept from deterministic to probabilistic physical laws. In the next two sections, it will be shown that time reversal invariance in deterministic laws does not have a self-evident probabilistic counterpart, a fact that has been the source of some confusion.[6]

4 Four probabilistic symmetries

A circumstance, previously only given modest attention, is the fact that the concept of *time asymmetry* is becoming considerably more complicated when trying to define it for stochastic processes. Since laws governing such processes do not describe a particular course of events, but rather a number of alternatives, it is not possible to apply directly the Definitions 3 and 4 given above.

The problem is tackled here through an analysis of the probabilities of the evolution between two arbitrary states, A and B, that constitute condition and outcome in a conditional probability. The concepts *future-directed* and *past-directed probabilities* are of central importance in this connection and are defined as follows:

Definition 5 *Future-directed probability* is conditional probability in which the conditional state precedes the outcome state:

$$p(B[t + \Delta t]|A[t]) = \frac{p(B[t + \Delta t] \wedge A[t])}{p(A[t])} .$$

Definition 6 *Past-directed probability* is conditional probability in which the outcome state precedes the conditional state:

$$p(B[t]|A[t + \Delta t]) = \frac{p(B[t] \wedge A[t + \Delta t])}{p(A[t])} .$$

[6]This confusion is well described by, for example, Gal-Or (1972), Savitt (1995), and Holster (2003a). Savitt (1995) explicitly describes several different versions of time reversal invariance, both deterministic and probabilistic.

These two concept can be formulated in more general terms, by specifying the conditional state and the outcome state within two *time intervals* rather than at two points in time.[7] However, the narrower definitions above are sufficient for the present analysis, while also having the advantage of being more in accordance with the starting points of other authors.[8]

It is an important first step to note that time directed conditional probabilities can be time reversed with respect to two successions:

- the time order of the condition and the outcome (which constitutes future-directed and past-directed probability), and
- the time order of the two states to be considered.

Combining these two alternatives makes it possible to formulate four probabilities of evolution between two non-simultaneous states, A and B. Since time reversal of order, according to Definition 3, implies time reversal of the states in question (again expressed by an superscript '^{T}'), the possible combinations are:

condition precede outcome, A before B: $p(B[t+\Delta t]|A[t])$;

condition precede outcome, B before A: $p(A^T[t+\Delta t]|B^T[t])$;

outcome precede condition, A before B: $p(A[t]|B[t+\Delta t])$;

outcome precede condition, B before A: $p(B^T[t]|A^T[t+\Delta t])$.

These four probabilities can, in their turn, be combined in six pairs of symmetry,[9] with the assumption that the probabilities express *laws*, and thereby are time translation invariant:[10]

Symmetry I: $p(B[t+\Delta t]|A[t]) = p(B^T[t]|A^T[t+\Delta t])$;

Symmetry II: $p(B[t+\Delta t]|A[t]) = p(A[t]|B[t+\Delta t])$;

[7] This more general definition will be used in forthcoming articles, e.g. Skoruppa (2022b).

[8] For example Holster (2003a) and Bacciagaluppi (2010).

[9] The somewhat illogical order of these symmetries is motivated by the authors ambition to attain a suitable numbering of the symmetries.

[10] Each symmetry can be expressed in different variants, dependent on where in time the time intervals between condition and outcome state are placed. Actually, *Symmetry I* is replaced by the similar symmetry

$$p(B[t+\Delta t]|A[t]) = p(B^T[t-\Delta t]|A^T[t])$$

from Section 10 and forwards for reasons that is presented in footnote 17. However, as long as the probabilities are assumed to be time translation invariant, this has no practical meaning.

Symmetry III: $p(B[t + \Delta t]|A[t]) = p(A^T[t + \Delta t]|B^T[t])$;

Symmetry IV: $p(A[t]|B[t + \Delta t]) = p(B^T[t]|A^T[t + \Delta t])$;

Symmetry Ib: $p(A^T[t + \Delta t]|B^T[t]) = p(A[t]|B[t + \Delta t])$;

Symmetry IIb: $p(A^T[t + \Delta t]|B^T[t]) = p(B^T[t]|A^T[t + \Delta t])$.

It is possible to reduce the number of unique time symmetries, as the notation above suggests. According to the premises, A and B are arbitrary states, and therefore the meaning of the symmetry is not affected if there is a shift from B to A^T simultaneously with a shift from A to B^T and vice versa in the six expressions. And since *Symmetry I* is identical to *Symmetry Ib*, if the names of the two states are shifted in this way, these two symmetries have the same meaning. For analogous reasons, *Symmetry II* and *Symmetry IIb* have the same meaning.

This correspondence between *Symmetries I* and *Ib* on the one hand, and *Symmetries II* and *IIb* on the other, does not exclude instances in which only one of the symmetries in each pair is valid (this is the case in the ant model in Section 7). In such cases, it can be concluded that the symmetry is not globally valid for the process in question, where *global validity* means that the symmetry is valid for all times and for every physically possible pair of states in the process.

As a result of a first comparison between the symmetries defined above, it becomes clear that *Symmetry I* and *Symmetry II* mirror a future-directed (left side of the equality sign) and a past-directed probability. *Symmetry III*, on the other hand, mirrors two future-directed probabilities, while *Symmetry IV* mirrors two past-directed probabilities.

5 The significance of the symmetries for deterministic physical laws

Now, the question is which of the four symmetries, given in the previous section, is the most suitable criterion for time reversal invariance for probabilistic laws. A reasonable condition is that it should coincide with the time reversal invariance in deterministic laws, given that all transition probabilities are either 0 or 1. The aim of this section is to test which of the symmetries can serve this purpose.

First, a bridge from mathematical to logical notation is needed. Generally, the following implication is valid under the assumption that A, B, C, and D are arbitrary states in a process governed by deterministic transition probabilities describing physical processes, where "xor" is the logical operator *either/or*:

$$
\begin{aligned}
&[p(B|A) = p(D|C) = 0] \text{ xor } [p(B|A) = p(D|C) = 1] \\
\Rightarrow\ &[p(B|A) = 0 \wedge p(D|C) = 0] \text{ xor } [p(B|A) = 1 \wedge p(D|C) = 1] \\
\Rightarrow\ &[(A \Rightarrow \neg B) \wedge (C \Rightarrow \neg D)] \text{ xor } [(A \Rightarrow B) \wedge (C \Rightarrow D)] \Rightarrow \{\text{Postulate 1}\} \\
\Rightarrow\ &[(A \Leftrightarrow \neg B) \wedge (C \Leftrightarrow \neg D)] \text{ xor } [(A \Leftrightarrow B) \wedge (C \Leftrightarrow D)] \\
\Rightarrow\ &[(A \Leftrightarrow B) \Leftrightarrow (C \Leftrightarrow D)]\ . \qquad (1)
\end{aligned}
$$

If the relation (1) is applied to the four symmetries *I*, *II*, *III*, and *IV* in the deterministic case, the following expressions are obtained:

Symmetry I

$$
\begin{aligned}
&p(B[t+\Delta t]|A[t]) = p(B^T[t]|A^T[t+\Delta t]) \Rightarrow \{\text{deterministic laws}\} \\
\Rightarrow\ &[p(B[t+\Delta t]|A[t]) = p(B^T[t]|A^T[t+\Delta t]) = 0] \text{ xor} \\
&\qquad [p(B[t+\Delta t]|A[t]) = p(B^T[t]|A^T[t+\Delta t]) = 1] \Rightarrow \{(1)\} \\
\Rightarrow\ &[(A[t] \Leftrightarrow B[t+\Delta t]) \Leftrightarrow (B^T[t] \Leftrightarrow A^T[t+\Delta t])]\ ; \qquad (2)
\end{aligned}
$$

Symmetry II

$$
\begin{aligned}
&p(B[t+\Delta t]|A[t]) = p(A[t]|B[t+\Delta t]) \Rightarrow \{\text{deterministic laws}\} \\
\Rightarrow\ &[p(B[t+\Delta t]|A[t]) = [p(A[t]|B[t+\Delta t]) = 0] \text{ xor} \\
&\qquad [p(B[t+\Delta t]|A[t]) = [p(A[t]|B[t+\Delta t]) = 1] \Rightarrow \{(1)\} \\
\Rightarrow\ &[(A[t] \Leftrightarrow B[t+\Delta t]) \Leftrightarrow (A[t] \Leftrightarrow B[t+\Delta t])]\ ; \qquad (3)
\end{aligned}
$$

Symmetry III

$$
\begin{aligned}
&p(B[t+\Delta t]|A[t]) = p(A^T[t+\Delta t]|B^T[t]) \Rightarrow \{\text{deterministic laws}\} \\
\Rightarrow\ &[p(B[t+\Delta t]|A[t]) = p(A^T[t+\Delta t]|B^T[t]) = 0] \text{ xor} \\
&\qquad [p(B[t+\Delta t]|A[t]) = p(A^T[t+\Delta t]|B^T[t]) = 1] \Rightarrow \{(1)\} \\
\Rightarrow\ &[(A[t] \Leftrightarrow B[t+\Delta t]) \Leftrightarrow (B^T[t] \Leftrightarrow A^T[t+\Delta t])]\ ; \qquad (4)
\end{aligned}
$$

Symmetry IV

$$
\begin{aligned}
&p(A[t]|B[t+\Delta t]) = p(B^T[t]|A^T[t+\Delta t]) \Rightarrow \{\text{deterministic laws}\} \\
\Rightarrow\ &[p(A[t]|B[t+\Delta t]) = p(B^T[t]|A^T[t+\Delta t]) = 0] \text{ xor} \\
&\qquad [p(A[t]|B[t+\Delta t]) = p(B^T[t]|A^T[t+\Delta t]) = 1] \Rightarrow \{(1)\} \\
\Rightarrow\ &[(A[t] \Leftrightarrow B[t+\Delta t]) \Leftrightarrow (B^T[t] \Leftrightarrow A^T[t+\Delta t])]\ . \qquad (5)
\end{aligned}
$$

The implications (2), (3), (4), and (5) can be summarized as follows:

> *Symmetries I*, *III*, and *IV* imply, when applied to deterministic physical laws, that $(A[t] \Leftrightarrow B[t+\Delta t]) \Leftrightarrow (A^T[t+\Delta t] \Leftrightarrow B^T[t])$; and

$$\textit{Symmetry I}: \left\{\begin{matrix} \textbf{Before} & \textbf{After} \\ A & B \\ \text{Condition} & \text{Outcome} \end{matrix}\right\} = \left\{\begin{matrix} \textbf{After} & \textbf{Before} \\ A & B \\ \text{Condition} & \text{Outcome} \end{matrix}\right\}$$

$$\textit{Symmetry II}: \left\{\begin{matrix} \text{Before} & \text{After} \\ A & B \\ \textbf{Condition} & \textbf{Outcome} \end{matrix}\right\} = \left\{\begin{matrix} \text{Before} & \text{After} \\ A & B \\ \textbf{Outcome} & \textbf{Condition} \end{matrix}\right\}$$

$$\textit{Symmetry III}: \left\{\begin{matrix} \text{Before} & \text{After} \\ \boldsymbol{A} & \boldsymbol{B} \\ \text{Condition} & \text{Outcome} \end{matrix}\right\} = \left\{\begin{matrix} \text{Before} & \text{After} \\ \boldsymbol{B} & \boldsymbol{A} \\ \text{Condition} & \text{Outcome} \end{matrix}\right\}$$

$$\textit{Symmetry IV}: \left\{\begin{matrix} \text{Before} & \text{After} \\ \boldsymbol{A} & \boldsymbol{B} \\ \text{Outcome} & \text{Condition} \end{matrix}\right\} = \left\{\begin{matrix} \text{Before} & \text{After} \\ \boldsymbol{B} & \boldsymbol{A} \\ \text{Outcome} & \text{Condition} \end{matrix}\right\}$$

Figure 1: Schematic representation of the four symmetries.

> *Symmetry II* implies, when applied to deterministic physical laws, that $(A[t] \Leftrightarrow B[t+\Delta t]) \Leftrightarrow (A[t] \Leftrightarrow B[t+\Delta t])$.

The former expression is identical to the criterion in Definition 4. Accordingly, Symmetries *I*, *III*, and *IV*, when applied to deterministic physical laws, coincide with time reversal invariance, which, at the beginning of this section was set as a condition for probabilistic time reversal invariance. The latter of these two expressions is a tautology and *Symmetry II*, therefore, does not represent any symmetry of physical significance in the deterministic case. Thus, the analysis in this section does not decide which of the three symmetries, *I*, *III*, and *IV*, ought to be the criterion for probabilistic time reversal invariance. However, *Symmetry II* is excluded for this aim.

6 What the symmetries mirror

To solve the question of which symmetry is best suited to constitute the criterion for probabilistic time reversal invariance, a possible procedure is to describe the four symmetries with regard to what is reversed and what is not. Such a description is given a schematic representation in Figure 1.

In this figure every probability is symbolized in braces by the three essential components: succession (before and after), state (A and B), and function in the conditional probability (condition and outcome). The left and right sides of the equalities are represented as similar to

Table 1: Interpretation of the four symmetries.

	Reversed:	Preserved:
I	Before and after	Condition state and outcome state
II	Condition and outcome	Which state is before and which is after
III	State A and state B	Condition comes before outcome
IV	State A and state B	Outcome comes before condition

each other as possible, in order to make clear what is reversed (bold letters) and what is preserved in each symmetry (see Table 1).

Symmetry I seems to imply true time symmetry according to the interpretation in Table 1. This can be proven through a derivation of the probabilistic time symmetry by applying the transformation $T : t \to -t$ to the left-hand side of the equation representing *Symmetry I*: [11]

$$\begin{aligned} & p(B[t+\Delta t]|A[t]) = T[p(B[t+\Delta t]|A[t])] \\ \Rightarrow\ & p(B[t+\Delta t]|A[t]) = p(B^T[-t-\Delta t]|A^T[-t]) \\ \Rightarrow\ & \{\text{time translation invariance}\} \\ \Rightarrow\ & p(B[t+\Delta t]|A[t]) = p(B^T[t]|A^T[t+\Delta t])\ . \end{aligned} \tag{6}$$

The last line of (6) is identical to *Symmetry I* and this symmetry can therefore, for good reasons, be regarded as the probabilistic counterpart of the criterion for time reversal invariance being described in Definition 3. None of *Symmetries II*, *III* or *IV* expresses this kind of symmetry with respect to time reversal according to the transformation $T : t \to -t$.

Symmetry II solely mirrors the condition and outcome of the probability. In agreement with the conclusions in the previous section it should therefore be excluded as a candidate for the role as probabilistic time reversal invariance criterion.

Symmetries III and *IV* mirror, according to Table 1, the *succession* of states represented in the probability, rather than the time direction with regard to before and after. In other words, *III* and *IV* are symmetries that mirror the time order of the process rather than time itself. Furthermore, these two symmetries appear to be reflections of each other, as both of them, according to Table 1, mirror the succession of the states on the basis of future-directed and past-directed probabilities, respectively. The fact that *Symmetry III* and *Symmetry IV* are time reversals of each other can formally be proven, by means of the

[11] Holster (1990) makes a derivation of what he calls Criterion for Probabilistic Reversibility in a similar way.

transformation $T : t \to -t$, applied to either of them:

$$\begin{aligned}
& T[\textit{Symmetry III}] \\
\Rightarrow\ & T[p(B[t+\Delta t]|A[t])] = T[p(A^T[t+\Delta t]|B^T[t])] \\
\Rightarrow\ & p(B^T[-t-\Delta t]|A^T[-t]) = p(A[-t-\Delta t]|B[-t])] \\
\Rightarrow\ & \{\text{time translation invariance}\} \\
\Rightarrow\ & p(B^T[t]|A^T[t+\Delta t]) = p(A[t]|B[t+\Delta t])\ ,
\end{aligned}$$

which is identical to *Symmetry IV*.

As *Symmetries III* and *IV* are time reversals of each other but not identical, they are themselves time asymmetric. Therefore, neither *Symmetry III* nor *Symmetry IV* represents a symmetry between a probability and its time reversal counterpart, since such symmetry must itself be time symmetric.

7 The ant model

The characteristics of the four symmetries analysed above can be made even clearer through a simple model. Suppose a confused ant is crawling around in a very big tree, in which all of the branches ramify in two directions for each metre measured vertically. For every metre upwards, therefore, the number of branches doubles. Each time the ant arrives at a point of ramification, it randomly chooses one of the three possible branches (upwards left, upwards right, or downwards). The choice of route is always made according to the same probabilities, which therefore prove to be time translation invariant.

As this is a model, and not a real tree, it can be assumed to be infinitely large, both upwards and downwards, whereby it is justified to ignore the possibility that the ant would reach an upper or lower limit (the top or the root). Moreover, it is assumed that the ant once was born or was placed in the tree with equal probability for each ramification point, and that it has been moving according to time translation invariant probabilities since then. Since every ramification point thereby has the same status in all respects, the unconditional probabilities for all the points will remain equal, though infinitesimal small.

Let $n[t]$ denote that the ant is situated at a branching point at an arbitrary level n at the point in time t, and let increasing n denote higher tree levels. Assume that crawling between two branching points requires one unit of time. Given this, it is possible to formulate four probabilities with respect to the ant moving upwards or downwards a *single branch* (Figure 2).[12]

[12] If the probabilities were described with respect to moves between *levels* rather

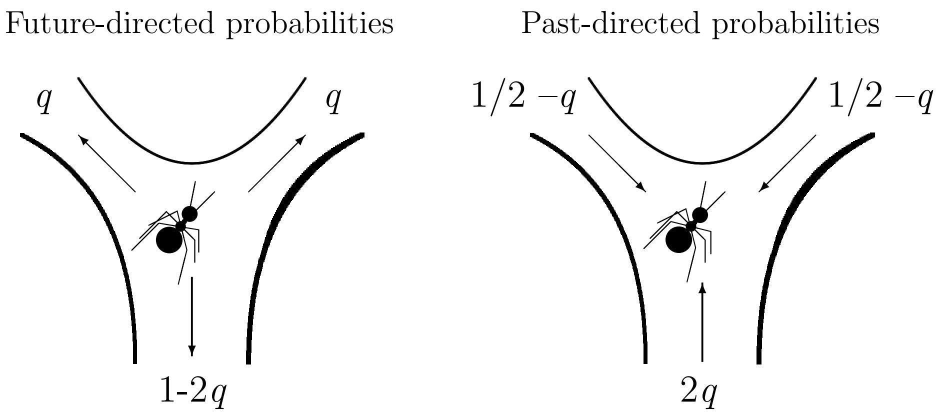

Figure 2: The future- and past-directed probabilities of the ant model.

For the sake of simplicity the two upper probabilities are assumed to be equal to each other, both in the future-directed case and in the past-directed case. Hence, the two possible future routes, given a position at an arbitrary branching point – *to the left branch* and *to the right branch* – are assumed to have equal future-directed probabilities q. Consequently, the remaining possibility *to the branch below* must have the probability $1 - 2q$, for the sum of all three to equal 1.

The corresponding past-directed probabilities can now be revealed by deduction. The probability of moving upwards must be independent of which kind of time-directed probabilities that describe the process, and since the unconditional probabilities are equal, the sum of the two future-directed probabilities of moving upwards, $2q$, must equal the past-directed probability of coming from a lower branch. And since the conditions imply that there is no difference between left and right, the remaining two past-directed probabilities, coming from the upper left and from the upper right, must be $1/2 - q$ for the sum of the past-directed probabilities to equal 1. In summary,

Future-directed probabilities

$$p(n+1[t+1]|n[t]) = q \ ;$$

$$p(n[t+1]|n+1[t]) = 1 - 2q \ ;$$

Past-directed probabilities

$$p(n[t]|n+1[t+1]) = 2q \ ;$$

$$p(n+1[t]|n[t+1]) = 1/2 - q \ .$$

than to *single branches*, $q = 1/2$ would be required for *Symmetries I*, *III* and *IV* to be valid.

These probabilities can be applied to the symmetries analysed in the preceding sections, and all six symmetries in the first formulation in Section 4 are expressed below to test under which conditions the symmetries are globally valid in the ant model. For arbitrary t, these conditions are (no superscript '^{T}' is needed, since every state is defined as its own time reversal)

Symmetry I

$$p(n+1[t+1]|n[t]) = p(n+1[t]|n[t+1]) \Leftrightarrow q = 1/2 - q \Leftrightarrow \; q = 1/4 \;;$$

Symmetry II

$$p(n+1[t+1]|n[t]) = p(n[t]|n+1[t+1]) \Leftrightarrow q = 2q \Leftrightarrow q = 0 \;;$$

Symmetry III

$$p(n+1[t+1]|n[t]) = p(n[t+1]|n+1[t]) \Leftrightarrow \; q = 1 - 2q \Leftrightarrow q = 1/3 \;;$$

Symmetry IV

$$p(n[t]|n+1[t+1]) = p(n+1[t]|n[t+1]) \Leftrightarrow 2q = 1/2 - q \Leftrightarrow \; q = 1/6 \;;$$

Symmetry Ib

$$p(n[t+1]|n+1[t]) = p(n[t]|n+1[t+1]) \Leftrightarrow 1 - 2q = 2q \Leftrightarrow q = 1/4 \;;$$

Symmetry IIb

$$p(n[t{+}1]|n+1[t]) = p(n+1[t]|n[t{+}1]) \Leftrightarrow 1 - 2q = 1/2 - q \Leftrightarrow q = 1/2 \;.$$

On the basis of these calculations, the conclusion can be drawn that *Symmetry II* is not globally valid in this model. It follows from the fact that *Symmetry II* and *Symmetry IIb* must both hold for *Symmetry II* to be globally valid, however they have incompatible implications, since $0 \neq 1/2$. On the other hand, *Symmetry I* is globally valid if $q = 1/4$, *Symmetry III* is valid if $q = 1/3$, and *Symmetry IV* is valid if $q = 1/6$. Hence, each symmetry requires different conditions for this model, except for *Symmetry II*, for which there is no condition to make it globally valid.

According to the analysis in Section 6, only the validity of *Symmetry I* necessarily requires a time symmetric evolution. This conclusion is verified in the ant model. If *Symmetry I* is globally valid, the ant can be expected to on average remain at the same level, since the probabilities of moving upwards equal the probabilities of moving downwards in both the future-directed and the past-directed case.

As already stated, *Symmetry II* can never give a globally valid description of the ant model. *Symmetry III*, in contrast, implies that the ant on average moves one level upwards every third unit of time. *Symmetry IV* implies an expected movement downwards of the same average speed as for *Symmetry III* but in the opposite direction. So in this model, when *Symmetry III* or *IV* is valid, the ant is not moving symmetrically with respect to time. To sum up, the ant is moving time symmetrically if and only if *Symmetry I* is valid and *Symmetry II*, *III*, and *IV* is not.

Taken together, this justifies the general conclusion that *Symmetry II*, *Symmetry III* and *Symmetry IV* are neither necessary nor sufficient criteria for time symmetric evolution.[13] On the other hand, *Symmetry I* is both a necessary and a sufficient criterion for time symmetric evolution in this model.

8 The clock model

It is also possible to construct a stochastic process for which only *Symmetry II* is valid, while none of the other three symmetries is. A simple evolution of this kind can be obtained in the following way.

A. Set a nonworking clock at 12 o'clock and throw an unbiased die. If the die shows 1 or 2, move the minute-hand twenty minutes backwards; if the die shows 5 or 6, move the minute-hand twenty minutes forwards (i.e. clockwise).

B. Throw the die again. If it shows 1 or 2, move the minute-hand twenty minutes backwards; if it shows 3, 4, 5 or 6, move the minute-hand twenty minutes forwards.

C. Repeat step B every 400 seconds (i.e. clockwise).[14]

The evolution of the system will now be described with respect to the number at which the *minute-hand* points. A series progressing according to the rules B and C can, for example, have the following appearance:

> 4, 8, 4, 8, 12, 4, 12, 4, 8, 12, 4, 12, 4, 8, 12, 8, 4, 8, 4, 12, 4, 8, 12, 4, 12.

The clock has, according to this example, in sum moved three turns clockwise during the first twenty-five moves.

[13] Holster (2003a) concludes from analysing a model equivalent to the ant model that *Symmetry III* (by Holster called "the orthodox criterion for time reversal invariance") is not a sufficient criterion for time symmetric evolution.

[14] The time interval between each step of the model is chosen to make the clock have an expected outcome equal to that of an ordinary clock (every 400 seconds). However, such a clock is a bad time-keeper, since it in the long run measures the time interval from the initial state with an expected ever decreasing relative error.

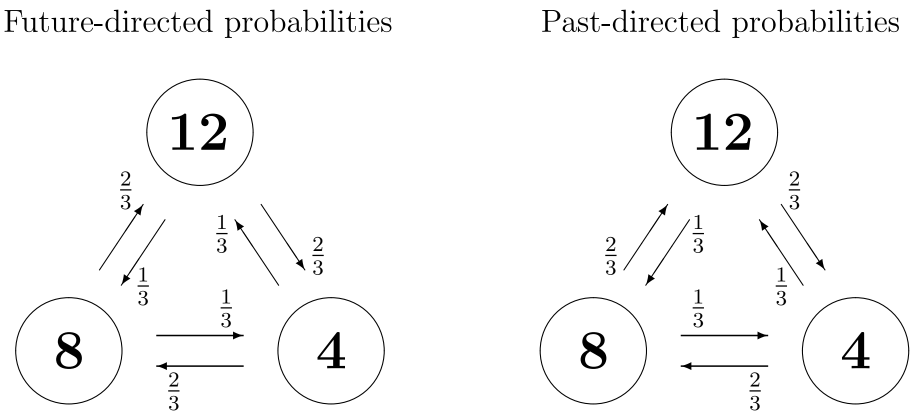

Figure 3: The future-directed and past-directed probabilities of the clock model (the arrows show the direction of the evolution and the values are placed close to the outcome).

In Figure 3 the future- and past-directed probabilities of a single step B of the model are presented (step A is not included in the description). Note that the two kinds of time-directed probabilities are equal for two given states with a given time order, which is the core characteristic of *Symmetry II.* The future-directed and past-directed probabilities can be expressed, each with one matrix, in the following way, where Δt is a positive integer multiple of 400 seconds.

Future-directed probabilities

$$\begin{bmatrix} p(4[t+\Delta t]|4[t]) & p(4[t+\Delta t]|8[t]) & p(4[t+\Delta t]|12[t]) \\ p(8[t+\Delta t]|4[t]) & p(8[t+\Delta t]|8[t]) & p(8[t+\Delta t]|12[t]) \\ p(12[t+\Delta t]|4[t]) & p(12[t+\Delta t]|8[t]) & p(12[t+\Delta t]|12[t]) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1/3 & 2/3 \\ 2/3 & 0 & 1/3 \\ 1/3 & 2/3 & 0 \end{bmatrix}^{\Delta t/400s} .$$

Past-directed probabilities

$$\begin{bmatrix} p(4[t-\Delta t]|4[t]) & p(4[t-\Delta t]|8[t]) & p(4[t-\Delta t]|12[t]) \\ p(8[t-\Delta t]|4[t]) & p(8[t-\Delta t]|8[t]) & p(8[t-\Delta t]|12[t]) \\ p(12[t-\Delta t]|4[t]) & p(12[t-\Delta t]|8[t]) & p(12[t-\Delta t]|12[t]) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/3 & 0 & 2/3 \\ 2/3 & 1/3 & 0 \end{bmatrix}^{\Delta t/400s} .$$

Thus *Symmetry II*,

$$p(B[t+\Delta t]|A[t]) = p(A[t-\Delta t]|B[t]) \ ,$$

is valid for arbitrary Δt and arbitrary states A and B in the model.

Symmetries I, *III*, and *IV*, on the contrary, are not valid for any combination of two different states in this model. As *Symmetry I* is not valid, the model ought to develop in a time asymmetric manner according to the analysis in Section 6. This can be confirmed by the future-directed (or past-directed) probabilities above, which imply that a clockwise evolution is more probable than an anticlockwise evolution:

$$\begin{aligned}
&\text{Expected clockwise move every 400 seconds}\\
&= p(\text{clockwise}[t+\Delta t]|\text{arbitrary}[t]) - p(\text{anticlockwise}[t+\Delta t]|\text{arbtirary}[t])\\
&= 2/3 - 1/3\\
&= 1/3 \ .
\end{aligned}$$

Since the evolution of the clock model thereby is time asymmetric, despite that *Symmetry II* is valid, the conclusion is that *Symmetry II* is not a sufficient criterion for a time symmetric evolution.

9 The thermodynamic equilibrium condition

Thermodynamic equilibrium is a condition of significant importance in the study of physical time symmetry. Since the state of equilibrium is macroscopically stable, it can be presumed to imply time symmetry, and therefore it is essential to find out under which conditions the four symmetries are valid in equilibrium. The present section is aimed at providing answers to this question.

Equilibrium is not really stationary in the case of a single factual evolution but rather consists of relatively small fluctuations around an exact equilibrium state. In order to make a statistical description of thermodynamic equilibrium, Willard Gibbs (1902) coined the concept of *statistical equilibrium* that is defined through an *ensemble*, which represents the distribution of probabilities of a macroscopically defined set of states at a given point in time. In equilibrium, this distribution is assumed to be stationary as long as the system is kept isolated.

It can be demonstrated that *Symmetry I* implies equilibrium, in the sense of constant unconditional non-zero probabilities, in a stochastic time-continuous process with a finite number of possible states, $\{S_1, S_2, \ldots, S_n\}$. This can be accomplished by first transforming *Symmetry I* into a more suitable version[15] (the superscripts '^{T}' are removed,

[15] Bacciagaluppi (2010) uses the symmetry $p(A[t+\Delta t]|B[t]) = p(A[t]|B[t+\Delta t])$

since the quantities defining a macroscopic thermodynamic state, i.e. volume, temperature, weight etc., are not influenced by time reversal):

$$
\begin{aligned}
& p(B[t'+\Delta t']|A[t']) = p(B[t']|A[t'+\Delta t']) \\
\Rightarrow & \{\text{time translation invariance}\} \\
\Rightarrow & p(B[t'+\Delta t']|A[t']) = p(B[t'-\Delta t']|A[t']) \\
\Rightarrow & \{\text{let } t' = t+\Delta t/2 \text{ and } \Delta t' = \Delta t/2\} \\
\Rightarrow & p(B[t+\Delta t]|A[t+\Delta t/2]) = p(B[t]|A[t+\Delta t/2]) \Rightarrow \{\text{definition}\} \\
\Rightarrow & \frac{p(B[t+\Delta t] \wedge A[t+\Delta t/2])}{p(A[t+\Delta t/2])} = \frac{p(B[t] \wedge A[t+\Delta t/2])}{p(A[t+\Delta t/2])} \\
\Rightarrow & p(B[t+\Delta t] \wedge A[t+\Delta t/2]) = p(B[t] \wedge A[t+\Delta t/2]) \; . \qquad (7)
\end{aligned}
$$

Next, it will be shown that *Symmetry I* is equivalent to detailed balance by demonstrating that implications between *Symmetry I* and detailed balance are valid in both directions. In a system with n possible states and with S_i, S_j and S_k denoting arbitrary states, the following relation holds, if the joint probabilities on the right side below are assumed to be independent of each other, i.e. if the process in question is assumed to be a Markov process:

$$p(S_i[t] \wedge S_k[t+\Delta t]) = \sum_{j=1}^{n} [p(S_i[t] \wedge S_j[t+\Delta t/2]) \cdot p(S_j[t+\Delta t/2] \wedge S_k[t+\Delta t])] \; .$$

Consequently, *Symmetry I* implies, through the application of equation (7) to both factors on the right side of this equality, that

$$
\begin{aligned}
& p(S_i[t] \wedge S_k[t+\Delta t]) \\
= & \sum_{j=1}^{n} [p(S_i[t] \wedge S_j[t+\Delta t/2]) \cdot p(S_j[t+\Delta t/2] \wedge S_k[t+\Delta t])] = (7) \\
= & \sum_{j=1}^{n} [p(S_i[t+\Delta t] \wedge S_j[t+\Delta t/2]) \cdot p(S_j[t+\Delta t/2] \wedge S_k[t])] \\
= & p(S_k[t] \wedge S_i[t+\Delta t]) \; . \qquad (8)
\end{aligned}
$$

This is an expression for *detailed balance*, which, in turn, implies equilibrium:

$$
\begin{aligned}
& p(S_i[t] \wedge S_k[t+\Delta t]) = p(S_k[t] \wedge S_i[t+\Delta t]) \\
\Rightarrow & \sum_{k=1}^{n} p(S_i[t] \wedge S_k[t+\Delta t]) = \sum_{k=1}^{n} p(S_k[t] \wedge S_i[t+\Delta t]) \\
\Rightarrow & p(S_i[t]) = p(S_i[t+\Delta t]) \; . \qquad (9)
\end{aligned}
$$

---

without transforming it and reaches similar results. Holster (2003a) presents a simpler but incorrect proof based on the unproven premise $p(A[t] \wedge B[t+\Delta t]) = p(A[t+\Delta t] \wedge B[t])$.

Furthermore, it can be shown that detailed balance, defined by equations (8) and (9), implies *Symmetry I*, under the assumption of time translation invariant future-directed probabilities (the same relation follows if the past-directed probabilities are assumed to be time translation invariant):

$$\begin{aligned} p(A[t-\Delta t]|B[t]) &= \frac{p(A[t-\Delta t] \wedge B[t])}{p(B[t])} = \{(8),(9)\} \\ &= \frac{p(A[t] \wedge B[t-\Delta t])}{p(B[t-\Delta t])} \\ &= p(A[t]|B[t-\Delta t]) = \{\text{time translation invariance}\} \\ &= p(A[t+\Delta t]|B[t]) \ . \end{aligned}$$

Thus, it can be concluded that, in stochastic time-continuous Markov processes with a finite number of possible states where the future-directed (and past-directed) probabilities are time translation invariant, *Symmetry I* is equivalent to detailed balance and implies equilibrium. However, since detailed balance is not a necessary condition for equilibrium, which is possible to demonstrate in the clock model,[16] neither is *Symmetry I* a necessary condition for equilibrium.

As a consequence of the following equivalence, a globally valid *Symmetry II* implies equilibrium:

$$\begin{aligned} & p(B[t+\Delta t]|A[t]) = p(A[t]|B[t+\Delta t]) \Leftrightarrow \{\text{definition}\} \\ \Leftrightarrow\ & p(B[t+\Delta t] \wedge A[t])/p(A[t]) = p(A[t] \wedge B[t+\Delta t])/p(B[t+\Delta t]) \\ \Leftrightarrow\ & p(A[t]) = p(B[t+\Delta t]) \ , \end{aligned}$$

where $p(A[t]) \neq 0 \neq p(B[t+\Delta t])$ must hold to avoid division by zero. This implies, since A and B are arbitrary states and Δt is an arbitrary time interval, that the unconditional probabilities of each state in the system must be constant through time, which means equilibrium. However, in case *Symmetry II* is globally valid, it also implies that every state has the same unconditional probability, which is a much stronger condition than equilibrium, and which means that the validity of *Symmetry II* is dependent on how the states are categorized. Therefore, a globally valid *Symmetry II* is a sufficient but not a necessary condition for equilibrium, given that none of the involved states has a zero unconditional probability.

Knowing that a globally valid *Symmetry I* is equivalent to detailed balance, it can be concluded that neither *Symmetry III* nor *Symmetry IV* is a necessary or a sufficient condition for detailed balance, since these

[16] The position of the minute-hand in the clock model in Section 8 is an example of an equilibrium process (since the unconditional probabilities for each of the three possible positions are constant) without detailed balance.

symmetries are not valid when *Symmetry I* is globally valid in the ant model in Section 7. Furthermore, neither *Symmetry III* nor *Symmetry IV* is a sufficient condition for equilibrium, since they do imply time asymmetric evolutions when they are globally valid in the ant model in Section 7, irrespective of the condition that all ramification points have the same unconditional probability; therefore, if the ant has a given initial or final position, the unconditional probabilities will change with time, which is incompatible with equilibrium. Finally, neither *Symmetry III* nor *Symmetry IV* is a necessary condition for equilibrium, since they are not valid in the clock model in Section 8, which is in equilibrium (with respect to the minute hand).

In conclusion, if the number of possible states is finite, a globally valid *Symmetry I* is equivalent to detailed balance, which implies that this symmetry is a sufficient condition for equilibrium, while a globally valid *Symmetry II* always is a sufficient conditions for equilibrium. Neither *Symmetry I* nor *Symmetry II* is a necessary condition for equilibrium. A globally valid *Symmetry III* or *Symmetry IV* is neither necessary nor sufficient conditions for equilibrium.

10 Two definitions of probabilistic symmetry

The analysis up to now can be summarized with respect to the properties of the four stochastic symmetries, given that the probabilities are time translation invariant (Table 2). Most important, the only symmetry that actually is an equality between a probability expression and its time reversal, according to implication (6) in Section 6, is *Symmetry I*. As can be seen in Table 2, *Symmetry I* is also the best choice with respect to other reasonable demands for the criterion of time reversal invariance considered in the previous sections. It is therefore well motivated to formulate a definition of probabilistic time reversal invariance based on a somewhat modified version of *Symmetry I* (similar to the second row of implication (7)):[17]

[17] Bacciagaluppi (2010) uses the symmetry $p(A[t+\Delta t]|B[t]) = p(A[t]|B[t+\Delta t])$. However, if the probabilities are not assumed to be time translation invariant, Bacciagaluppi's version of *Symmetry I* allows for time asymmetric transition probabilities that can be continuously increasing or decreasing in a Markov process, e.g. $p(A[t+1]|B[t]) = p(A[t]|B[t+1]) = 2^{-t}$, $t < 0$, for some arbitrary states A and B in a process with discrete-time evolution. The symmetry $p(A[t+\Delta t]|B[t]) = p(A[t]|B[t+\Delta t])$ is therefore a weaker criterion for time symmetry than $p(A[t+\Delta t]|B[t]) = p(A[t-\Delta t]|B[t])$, which is a necessary condition for momentary time symmetry, e.g. when B is a local entropy maximum or minimum during a fluctuation. The version that is used in Definition 7 is also in accordance with Holster's (2003a) "correct criterion for time reversal invariance".

Table 2: Time symmetry properties of the four stochastic symmetries.

Property	Section	I	II	III	IV
Corresponds with deterministic time reversal invariance.	5	yes	no	yes	yes
Expresses symmetry in accordance with the transformation $T : t \to -t$.	6	yes	no	no	no
Necessary condition for time symmetry in the ant model.	7	yes	no	no	no
Sufficient condition for time symmetry according to ant model and clock model.	7, 8	yes	no	no	no
Sufficient condition for equilibrium in a system with a finite number of states.	9	yes	yes	no	no
Necessary condition for equilibrium.	9	no	no	no	no
Equivalent to detailed balance in a system with a finite number of states.	9	yes	no	no	no

Definition 7 (**criterion for probabilistic time reversal invariance**)
A probabilistic law, L_p, is *time reversal invariant*, i.e. symmetric under the transformation $T : t \to -t$, if, for arbitrary states A and B as well as for all t,

$$p(B[t + \Delta t]|A[t]) = p(B^T[t - \Delta t]|A^T[t])$$

according to L_p.

Unfortunately, this conclusion has a considerable drawback. If the criterion for probabilistic time reversal invariance should be compatible with the current conventions, it must accord with the concept of T-symmetry, which is a part of the CPT symmetry, where C stands for charge conjugation, P for parity, and T for time reversal. The only direct violations of time reversal invariance that have been detected up to now are observed in the decays of the neutral K-meson and the B-meson.[18]

In accurate experiments it has been shown that the neutral K-meson has a somewhat greater probability to decay from particle to antiparticle than in the opposite direction. This is described as a violation of time reversal invariance, and the asymmetry can be mathematically expressed in the following way:

$$p(\textit{anti-}K^0[t + \Delta t]|K^0[t]) > p(K^0[t + \Delta t]|\textit{anti-}K^0[t]) \ .$$

By comparison with the four symmetries, it becomes clear that this is a violation of *Symmetry III* (a more thorough analysis of the decay of

[18] Angelopoulos et al. (1998) and Lees et al. (2012), respectively. The K-meson is also called *kaon*.

the neutral K-meson is given in Skoruppa (2022a)). The '^{T}' for the time reversal of the state is unnecessary, since the transformation between K^0 and *anti*-K^0 is not dependent on anything except which of those two states the particles are in.

The T-symmetry was introduced in the probabilistic context of quantum mechanics in 1932,[19] and the decay of the neutral K-meson, since 1964, has been considered to demonstrate violation of time reversal invariance.[20] Hence there is a long history of association between this type of invariance and *Symmetry III*,[21] and it seems reasonable to use this symmetry in the definition of time reversal invariance if there is not a very strong argument for doing otherwise.

Certainly, two major objections can be raised against defining *Symmetry III* as the criterion for probabilistic time reversal invariance. First, it stands against the convention that time reversal invariance should not mean symmetric under the transformation $T : t \to -t$. Second, considering it was demonstrated in Section 6 that *Symmetry III* is the time reversal of *Symmetry IV*, it is quite troublesome to have a criterion for time reversal invariance that itself is not time symmetric, i.e. is dependent on which time direction it is applied to. A related problem, for those who want to keep *Symmetry III* as the criterion for time reversal invariance, is to decide what kind of invariance *Symmetry IV* represents.

On the other hand, the issue of breaking *Symmetry III* is of such big importance in physics that the symmetry ought to have a name. A solution to this problem is to adopt a term that has been hinted by at least two well-reputed authors,[22] and thereby rename the symmetry broken in the decay of the neutral K-meson:[23]

> **Definition 8 (criterion for future-directed process reversal invariance)**
> A probabilistic law, L_p, is *future-directed process reversal invariant* if, for arbitrary states A and B as well as for all t,
> $$p(B[t + \Delta t]|A[t]) = p(A^T[t + \Delta t]|B^T[t])$$
> according to L_p.

[19] Wigner (1932).

[20] Christenson et al. (1964).

[21] Holster (2003b) describes in a concluding footnote how difficult it is to convince the scientific community to abandon *Symmetry III* as a criterion for time reversal invariance.

[22] Bunge (1972, p. 127) writes: "In short, *time reversal corresponds to process reversal.*" The same opinion is expressed by Davies (1974).

[23] This is called *the orthodox criterion for reversal symmetry* by Holster (2003a). In Skoruppa (2022c), *criterion for entropy equality* is proposed as an even more suitable name for this symmetry.

The name chosen here for this criterion is also in accordance with the analysis of *Symmetry III* in Section 6.

11 The physical significance of the symmetries

From the analyses of the four symmetries in the preceding sections, the conclusion can be drawn that they differ with respect to physical significance. In this section, some brief descriptions are given of their theoretical significance and how they have been used in previous analyses of physical processes.

Symmetry I is of great theoretical interest, since it actually describes probabilistic time reversal symmetry, i.e. symmetry under the transformation $T : t \to -t$. This symmetry has also been proposed by other authors as a criterion for time symmetry in stochastic processes.[24] As is demonstrated in Sections 7–9, this symmetry is globally valid in stable stochastic processes, where there is no directed evolution. In thermodynamics this means detailed balance, which implies equilibrium. Similar stable states for which *Symmetry I* is valid can also be found in a quantum mechanics, when, for example, there is balance between the outgoing and incoming particles in a system of radioactive atoms. *Symmetry I* can also be momentarily valid at points in time representing the local entropy minima of non-equilibrium in statistical mechanics, but then the symmetry is not time translation invariant.

Symmetry II has previously been given minor attention in physics, and the author of the present article has found it explicitly analysed only by Holster (2003a), who calls it "cause–effect exchange operator" and attributes very limited theoretical interest to it. Watanabe (1955) uses this symmetry when examining the "retrodictability" of quantum mechanics. In a later work Watanabe (1965) compares the applicability of two probabilities $p(F_j|E_i)$ and $p(E_i|F_j)$, where E_i happens before F_j, but this seems rather to be a comparison between the applicabil-

[24] An explicit proposal that *Symmetry I*, in the version of Definition 7, should serve as the criterion for time symmetry can be found in Holster (2003a), who suggests it as "the correct criterion for time reversal invariance". Arntzenius (1995) proposes, with a less rigorous notation, the same symmetry as a criterion for a probabilistic theory being "time-symmetric or time-reversible".

Uffink (2007) and Bacciagaluppi (2010) suggest the symmetry (notation from the present article)

$$p(A[t + \Delta t]|B[t]) = p(A[t]|B[t + \Delta t]) ,$$

which they describe as the criterion for the condition "that a stochastic process is reversible" and as a "condition for a time-symmetric process", respectively. This symmetry, however, is a weaker criterion for time symmetry (see footnote 17).

Healey (1981) and Callender (2000) can be interpreted as expressing opinions similar to those of the above mentioned authors. However, as a consequence of their obscure notations, it is not clear which symmetry they actually mean.

ity of future-directed and past-directed probabilities in general than an analysis of *Symmetry II*. Penrose (1989) compares two conditional probabilities $p(P[t+\Delta t]|L[t])$ and $p(L[t]|P[t+\Delta t])$ in a thought experiment, where a lamp, L, sends a photon through a half-silvered mirror to be detected in a photocell, P. Again, the aim is to compare the applicability of future-directed and past-directed probabilities rather than to scrutinize *Symmetry II*.[25] A reason for the limited significance of *Symmetry II* is that its validity is dependent on how the states are categorized (see Section 9).

Symmetry III has been given a lot of attention in physics, and it has been asserted to be the statistical version of the so called T-symmetry. When it was found that it is broken in the asymmetric decay of the neutral K-meson, it even brought a Nobel prize to the discoverers, and it has been established that it is broken also in other cases of weak interaction.[26] These findings have given rise to the hypothesis that this broken asymmetry can explain why there is a dominance of matter over anti-matter in our universe.[27] The violation of *Symmetry III* has furthermore been asserted to reveal a foundational time asymmetry in quantum mechanics.[28] This assertion, however, is false since *Symmetry III* is neither a necessary nor a sufficient criterion for time symmetry. The asymmetry of the neutral K-meson decay would be observed, even if there were just as many decays $K^0 \rightarrow$ anti-K^0 as anti-$K^0 \rightarrow K^0$, in a process thereby time symmetric. Hence, it is a mistake, which has been made by several renowned physicists, to interpret the behaviour of the neutral K-meson as a sign of nature differentiating between the past and future time directions.[29] In Skoruppa (2022a) doubt is also raised about this symmetry having general validity in quantum mechanical decay processes, even outside the realm of weak interaction.

Symmetry IV, finally, in accordance with the analysis in Section 6 in combination with Definition 8, can simply be described as a past-directed counterpart to *Symmetry III* and, therefore, suitably is named *past-directed process reversal invariance.* This symmetry demonstrates

[25] A physical law, based on such a test between future-directed and past-directed probabilities, is proposed in Skoruppa (2022b).

[26] Christenson et al. (1964), Lees et al. (2012).

[27] The idea that the breaking of *Symmetry III* of the weak interaction is the underlying cause of the dominance of matter over anti-matter in our universe was first suggested by Sakharov (1967). However, recent research seems to shed some doubt to this hypothesis (Peskin, 2002; Ellis, 2003).

[28] For example, Gell-Mann & Hartle (1994), Davies (1995), Peach (1998), and Ellis (2003) write that the decay of the neutral K-meson shows a difference between future and past time direction. See also footnote 29.

[29] For references, see footnote 28. Watanabe (1965) briefly expresses doubt about this interpretation. Bohm et al. (2011, p. 2) write that "TA [time asymmetry] is not to be mistaken for time reversal non-invariance".

the time-directed nature of *Symmetry III* as its mirror image with respect to time. Nothing indicates that *Symmetry IV*, under any circumstances, is generally valid in physics, except when a system is in a state of equilibrium.

The relation between *Symmetry III* and *Symmetry IV* can, on the basis of the analysis in Section 6, be expressed in the following way.

> **Meta-symmetric time asymmetry**
> The symmetry $p(B[t+\Delta t]|A[t]) = p(A^T[t+\Delta t]|B^T[t])$ is, the decay of the neutral K-meson and the B-meson being the only observed exceptions, generally valid in quantum mechanics, while its time reversal (by the transformation $T: t \to -t$) $p(A[t]|B[t+\Delta t]) = p(B^T[t]|A^T[t+\Delta t])$ is not.

Since *Symmetry III* describes the relation between future-directed probabilities and *Symmetry IV* describes the relation between past-directed probabilities, this asymmetry indicates that the future-directed probabilities have some sort of priority over the past-directed probabilities in the physical description of our universe. Similar proposals have also been made by other authors, and an explicit formulation of this time asymmetry will be the object of a forthcoming article.[30]

12 Conclusions

An analysis of the symmetry combinations for conditional probabilities, with condition and outcome at different points in time, shows that it is possible to find four unique symmetries. Three of these symmetries coincide with the definition of time reversal invariance for deterministic processes, given that the probabilities of the symmetries equal 0 or 1. The subsequent analysis shows that the symmetry $p(B[t+\Delta t]|A[t]) = p(B^T[t-\Delta t]|A^T[t])$ best meets different reasonable demands for *probabilistic time reversal invariance* (see Table 2) and therefore is chosen as the criterion for this concept.

The symmetry $p(B[t+\Delta t]|A[t]) = p(A^T[t+\Delta t]|B^T[t])$ is according to the current conventions associated with what is termed *time reversal invariance* in particle physics. However, since this symmetry rather corresponds to the reversal of processes and consists of an equality between two future-directed probabilities, it is established as a criterion for *future-directed process reversal invariance.*

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[30]Works by Gibbs (1902), Watanabe (1965), Sober (1993) and Arntzenius (1995), describing the priority of future-directed probabilities, are discussed in Skoruppa (2022b), where a *law of statistical time asymmetry* is proposed as well. However, the general validity of the T-symmetry $p(B[t+\Delta t]|A[t]) = p(A^T[t+\Delta t]|B^T[t])$ is questioned in Skoruppa (2022a).

Moreover, the existence of a meta-symmetric time asymmetry has become evident. Future-directed process reversal invariance (as defined in the present article) is generally valid, with a single known exception in particle physics, while the time reversed counterpart lacks validity. This empirically founded fact indicates a fundamental time asymmetry in our universe.

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References

Angelopoulos, A., Apostolakis, A., Aslanides, E., Backenstoss, G., Bargassa, P., Behnke, O., Benelli, A., Bertin, V., Blanc, F., Bloch, P., Carlsson, P., Carroll, M., Cawley, E., Charalambous, S., Chertok, M. B., Danielsson, M., Dejardin, M., Derre, J., Ealet, A., Eleftheriadis, C., Faravel, L., Fetscher, W., Fidecaro, M., Filipčič, A., Francis, D., Fry, J., Gabathuler, E., Gamet, R., Gerber, H. J., Go, A., Haselden, A., Hayman, P. J., Henry-Couannier, F., Hollander, R. W., Jon-And, K., Kettle, P. R., Kokkas, P., Kreuger, R., Le Gac, R., Leimgruber, F., Mandić, I., Manthos, N., Marel, G., Mikuž, M., Miller, J., Montanet, F., Muller, A., Nakada, T., Pagels, B., Papadopoulos, I., Pavlopoulos, P., Policarpo, A., Polivka, G., Rickenbach, R., Roberts, B. L., Ruf, T., Santoni, C., Schäfer, M., Schaller, L. A., Schietinger, T., Schopper, A., Tauscher, L., Thibault, C., Touchard, F., Touramanis, C., Van Eijk, C. W. E., Vlachos, S., Weber, P., Wigger, O., Wolter, M., Zavrtanik, D., & Zimmerman, C. (1998). First direct observation of time-reversal non-invariance in the neutral-kaon system. *Physics Letters B*, 444, 43–51.

Arntzenius, F. (1995). Indeterminism and the direction of time. *Topoi*, 14, 67–81.

Bacciagaluppi, G. (2010). Probability and time symmetry in classical Markov processes. In M. Suárez (Ed.), *Probabilities, Causes and Propensities in Physics* (pp. 41–60). Dordrecht: Springer Netherlands.

Bohm, A. R., Gadella, M., & Kielanowski, P. (2011). Time asymmetric quantum mechanics. *Sigma*, 7, 086.

Bunge, M. (1972). Time asymmetry, time reversal, and irreversibility. In J. T. Fraser, F. C. Hüber, & G. H. Müller (Eds.), *The Study of Time* (pp. 122–130). Berlin: Springer-Verlag.

Callender, C. (2000). Is time ‘handed’ in a quantum world? *Proceedings of the Aristotelian Society*, 100, 247–269.

Christenson, J. H., Cronin, J. W., Fitch, V. L., & Turlay, R. (1964). Evidence for the 2π decay in the ${K_2}^0$ meson. *Physical Review Letters*, 13, 138–140.

Davies, P. C. W. (1974). *The Physics of Time Asymmetry.* London: Surrey University Press.

Davies, P. C. W. (1995). *About Time: Einstein’s Unfinished Revolution.* London: Viking.

Earman, J. (1974). An attempt to add a little direction to “the problem of the direction of time”. *Philosophy of Science*, 41, 15–47.

Ellis, J. (2003). Antimatter matters. *Nature*, 424, 631–634.

Gal-Or, B. (1972). The crisis about the origin of irreversibility and time anisotropy. *Science*, 176, 11–17.

Gell-Mann, M. & Hartle, J. B. (1994). Time symmetry and asymmetry in quantum mechanics and quantum cosmology. In J. J. Halliwell, J. Pérez-Mercader, & W. H. Zurek (Eds.), *Physical Origins of Time Asymmetry* (pp. 311–345). Cambridge: Cambridge University Press.

Gibbs, W. (1902). *Elementary Principles in Statistical Mechanics.* New York: C. Scribner’s Sons.

Healey, R. (1981). Statstical theories, quantum mechanics and the directedness of time. In R. Healey (Ed.), *Reduction, Time, and Reality* (pp. 99–127). Cambridge: Cambridge University Press.

Holster, A. (1990). Time flow and reversibility in a probabilistic universe. PhD thesis. Available at `https://philpapers.org/rec/HOLFA-3`.

Holster, A. (2003a). The criterion for time symmetry of probabilistic theories and the reversibility of quantum mechanics. *New Journal of Physics*, 5, 130.1–130.28.

Holster, A. (2003b). The time reversal invariance of classical electromagnetic theory: Albert versus Malament. *PhilSci Archive, ID-code 1475,* 1–24. Internet: `http://philsci-archive.pitt.edu/archive/00001475/`.

Lees et al. (2012). Observation of time reversal violation in the B^0 meson system. *Physical Review Letters*, 109, 1–7.

Liu, C. (1993). The arrow of time in quantum gravity. *Philosophy of Science*, 60, 619–637.

Peach, K. (1998). Time's broken arrow. *Nature*, 396, 407–408.

Penrose, R. (1989). *The Emperor's New Mind.* Oxford: Oxford University Press.

Peskin, M. (2002). The matter with antimatter. *Nature*, 419, 24–27.

Sakharov, A. D. (1967). Violation of CP symmetry, C asymmetry and baryon asymmetry of the universe. *JETP Letters*, 5, 24–26.

Savitt, S. F. (1995). Introduction. In S. F. Savitt (Ed.), *Time's Arrows Today* (pp. 1–19). Cambridge: Cambridge University Press.

Skoruppa, B. (2022a). Probabilistic time asymmetry in quantum mechanical processes. Appendix VI.

Skoruppa, B. (2022b). A proposed law for statistical time asymmetry. Appendix II.

Skoruppa, B. (2022c). Statistical entropy difference. Appendix VIII.

Sober, E. (1993). Temporally oriented laws. *Synthese*, 94, 171–189.

Uffink, J. (2001). Bluff your way in the second law of thermodynamics. *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics*, 32, 305–394.

Uffink, J. (2007). Compendium of the foundations of classical statistical physics. In I. J. Butterfield & J. Earman (Eds.), *Philosophy of Physics* (pp. 923–1074). Amsterdam: Elsevier.

Watanabe, S. (1955). Symmetry of physical laws. Part III. Prediction and retrodiction. *Reviews of Modern Physics*, 27, 179–186.

Watanabe, S. (1965). Conditional probability in physics. *Supplement of the Progress of Theoretical Physics*, 33–34, 135–160.

Wigner, E. P. (1932). Über die Operation der Zeitumkehr in der Quantenmekanik. *Nachrichten der Gesellschaft der Wissenschaften zu Göttingen Mathematisch-Physikalische Klasse*, (pp. 546–559).

Zeh, H.-D. (1992). *The Physical Basis of the Direction of Time.* Heidelberg: Springer-Verlag.